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Extensions of Semi-Closure Spaces

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1. Introduction

In this paper we study an extension of semi-closure spaces and some general properties of this extension.

Let X be any set and $P(X)$ the power set of X. A function $u: P(X) \to P(X)$ is called a ***semi-closure structure*** [3] on X if it satisfies the following four conditions:

i) $u(\phi)=\phi$,

ii) $A \subset u(A)$ for each $A \in P(X)$,

iii) $A \subset B \Rightarrow u(A) \subset u(B)$, for each $A, B \in P(X)$,

iv) $u(A)=u(u(A))$, for each $A \in P(X)$.

A pair (X, u) where u is a semi-closure structure on X, is called a *semi-closure space*. These concepts are generalizations of the more familiar Kuratowski closure operator and topological space, respectively. For a convinience, we shall agree to use u as $\{A \mid u(X-A)=X-A\}$. A set A is called a *semi-open* (resp. *semi-closed*) subset of (X, u) if $A \in u$ (resp. $X-A \in u$). Let f be a function from a semi-closure space (X, u) into a semi-closure space (Y, w). If for every $A \in w$, $f^{-1}(A) \in u$ (resp. for every $A \in u$, $f(A) \in w$, then we shall say f is *s-continuous* (resp. *s-open*). If for every semi-closed set A (i.e., $u(A)=A$), $w(f(A))=f(A)$ then we shall say f is *s-closed*. Moreover, a bijective function f is called an *s-homeomorphism* if f is *s-continuous* and *s-open* [See, 3].

Definition 1.1. *Let (X, u) be a semi-closure space. A collection $\mathscr{A}$ of subsets of X is called an u-bunch on X if it satisfies the following three conditions:*

i) $\phi \notin \mathscr{A}$ and $\phi \neq \mathscr{A}$,

ii) $A \in \mathscr{A}$ and $A \subset B \Rightarrow B \in \mathscr{A}$,

iii) $u(A) \in \mathscr{A} \Rightarrow A \in \mathscr{A}$.

The following lemma is easily established.

Lemma 1.2. *Let (X, u) be a semi-closure space. Then,*

(1) $\mathscr{A}_u(x)=\{A \subset X \mid x \in u(A)\}$ is an u-bunch on X for each $x \in X$. Moreover, for every $A \in P(X)$ $\mathscr{A}_u(A)=\{B \subset X \mid A \subset u(B)\}$ is an u-bunch on X.

(2) $\bigcap_{x \in A} \mathscr{A}_u(x)=\mathscr{A}_u(A)$, for exery $A \in P(X)$.

(3) $u(A)=\{x \in X \mid A \in \mathscr{A}_u(x)\}$ for every $A \in P(X)$.

From the above definition and lemma, we show that the collection $\{\mathscr{A}_u(x) \mid x \in X\}$ of u-bunches of (X, u) determined u completly just as u determined all $\mathscr{A}_u(x)$.

2. Extensions of semi-closure spaces

Let $f: X \to Y$ be an injection; let u be a semi-closure structure on X an k be a semi-closure structure on Y. Then $(f, (Y, k))$ is called an *extension* [2] of (X, u) if

i) $k(f(X)) = Y$ and,

ii) $f(u(A)) = k(f(A)) \cup f(X)$ for every $A \in P(X)$.

Since f is injective, ii) insures that f is s-homeomorphism from (X, u) onto $(f(X), k')$, where $k'(B) = k(B) \cup f(X)$ for every $B \subset f(X)$, is the semi-closure structure induced on $f(X)$ [3] by the semi-closure structure k on Y. Condition i) insures that $f(X)$ is dense in (Y, k).

A semi-closure (X, u) shall be called an *S_0-space* if

$$\mathscr{A}_u(x) = \mathscr{A}(y) \Rightarrow x = y.$$

We are now able to state and prove an S_0-extension of an S_0-space (X, u).

Theorem 2.1. *Let (X, u) be an S_0-space. Let X^* be the set of all u-bunches on X. Define*

$k_u : P(X^*) \to P(X^*)$ by $k_u(\alpha) = \{\mathscr{A} \in X^* \mid \cap \alpha \subset \mathscr{A}\}$ for each $\alpha \in P(X^*)$,

$\varphi_X : X \to X^*$ by $\varphi_X(x) = \mathscr{A}_u(x)$ for each $x \in X$.

Then $(\varphi_X, (X^*, k_u))$ is an S_0-extension of (X, u).

Proof. First we show that k_u is a semi-closure structure on X^*.

i) Let $\mathscr{A} \in X^*$. Then $\phi \notin \mathscr{A}$ and $\cap \phi \equiv P(X) \not\subset \mathscr{A}$. Thus $\mathscr{A} \notin k_u(\phi)$, that is, $k_u(\phi) = \phi$.

ii) Let $\alpha \in P(X^*)$. If $\mathscr{A} \in \alpha$, $\cap \alpha \subset \mathscr{A}$ and so $\mathscr{A} \in k_u(\alpha)$. Thus, $\alpha \subset k_u(\alpha)$.

iii) Let α and β be two elements of $P(X^*)$ with $\alpha \subset \beta$. If $\mathscr{A} \in k_u(\alpha)$, $\cap \alpha \subset \mathscr{A}$ and $\cap \beta \subset \mathscr{A}$, since $\cap \beta \subset \cap \alpha$. Thus, $\mathscr{A} \in k_u(\beta)$, that is, $k_u(\alpha) \subset k_u(\beta)$.

iv) We shall show that $k_u(k_u(\alpha)) \subset k_u(\alpha)$ for every $\alpha \in P(X^*)$. Suppose that there exists an $\mathscr{A} \in X^*$ such that $\mathscr{A} \in k_u(k_u(\alpha))$ and $\mathscr{A} \notin k_u(\alpha)$. Then $\cap \alpha \not\subset \mathscr{A}$, there exists $A \in \cap \alpha$ such that $A \notin \mathscr{A}$. Since $\mathscr{A} \in k_u(k_u(\alpha)) \Leftrightarrow \mathscr{A} \supset \cap k_u(\alpha)$, $A \notin \cap k_u(\alpha)$. There exists $\mathscr{A}'$ in $k_u(\alpha)$ such that $A \notin \mathscr{A}'$. Therefore, $A \in \cap \alpha \subset \mathscr{A}'$ and $A \notin \mathscr{A}'$, a contradiction.

By i), ii), iii), and iv), k_u is a semi-closure on X^*.

Next, we show that $(\varphi_X, (X^*, k_u))$ is an extension of (X, u). Clearly, φ_X is an injection into X^*. Moreover,

$$\begin{aligned} k_u(\varphi_X(X)) &= k_u(\{\mathscr{A}_u(x) \mid x \in X\}) \\ &= \{\mathscr{A} \in X^* \mid \bigcap_{x \in X} \mathscr{A}_u(x) \subset \mathscr{A}\} \\ &= \{\mathscr{A} \in X^* \mid \mathscr{A}_u(X) \subset \mathscr{A}\}, \text{ by Lemma 1.2,} \\ &= X^* \end{aligned}$$

so that $\varphi_X(X)$ is dense in X^*.

For each subset A of X,

$$\begin{aligned} \varphi_X(u(A)) &= \varphi_X(\{x \in X \mid A \in \mathscr{A}_u(x)\}) \\ &= \{\mathscr{A}_u(x) \mid A \in \mathscr{A}_u(x)\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} k_u(\varphi_X(A)) \cap \varphi_X(X) &= k_u(\{\mathscr{A}_u(x) \mid x \in A\}) \cup \varphi_X(X) \\ &= \{\mathscr{A} \in \varphi_X(X) \mid \bigcap_{x \in A} \mathscr{A}_u(x)\} \end{aligned}$$

$$=\{\mathscr{A}_u(x)\,|\,A\in\mathscr{A}_u(x)\},\ \text{by Lemma 1.2.}$$

Thus, $\varphi_X(u(A))=k_u(\varphi_X(A))\cap\varphi_X(X)$ for each $A\in P(X)$.

Finally, we show that (X^*, k_u) is an S_0-space. Define $A_{k_u}(\mathscr{A})=\{\mathfrak{B}\in X^*\,|\,\mathscr{A}\in k_u(\mathfrak{B})\}$ for each element $\mathscr{A}$ of X^*. Suppose that $A_{k_u}(\mathscr{A})=A_{k_u}(\mathfrak{B})$ and $\mathscr{A}\neq\mathfrak{B}$ (*i.e.*, $\mathscr{A}\not\supset\mathfrak{B}$ or $\mathscr{A}\not\subset\mathfrak{B}$). Then, if $\mathscr{A}\not\supset\mathfrak{B}$, then $\{\mathscr{A}\}\notin A_{k_u}(\mathfrak{B})$ but $\{\mathscr{A}\}\in A_{k_u}(\mathscr{A})$. Similarly, $\{\mathfrak{B}\}\notin A_{k_u}(\mathscr{A})$ and $\{\mathfrak{B}\}\in A_{k_u}(\mathfrak{B})$. Thus, $A_{k_u}(\mathscr{A})\neq A_{k_u}(\mathfrak{B})$, a contradiction.

Therefore, $(\varphi_X, (X^*, k_u))$ is an S_0-extension of (X, u).

We now investigate some general properties of the extension $(\varphi_X, (X^*, k_u))$ of (X, u) which is similar to the case of extensions of nearness spaces [1] and the case of extensions of closure spaces [2].

Theorem 2.2. *Let (X, u) and (Y, w) be two S_0-spaces and let $f: X\to Y$ be function. Define f^*: $(X^*, k_u)\to(Y^*, k_w)$ by*

$$f^*(\mathscr{A})=\{B\subset Y\,|\,f^{-1}(w(B))\in\mathscr{A}\}\ \textit{for each}\ \mathscr{A}\in X^*,$$

where $(\varphi_X, (X^*, k_u))$ and $(\varphi_Y, (Y^*, k_w))$ are the extension of (X, u) and (Y, w), respectively. Then,

(1) f^* is a function,

(2) $f(\mathscr{A})\subset f^*(\mathscr{A})$ for each $\mathscr{A}\in X^*$.

(3) If f is s-continuous and s-closed, then $f^*(\mathscr{A}_u(x))=\mathscr{A}_w(f(x))$ for each $x\in X$ and $f^*\circ\varphi_X=\varphi_Y\circ f$.

(4) If f is s-continuous and s-closed, then f^* is s-continuous.

Proof. (1) We shall show that every image of an u-bunch on X is a w-bunch on Y, that is, $f^*(\mathscr{A})\in Y^*$ for each $\mathscr{A}\in X^*$.

i) $\phi\notin f^*(\mathscr{A})$ and $\phi\neq f^*(\mathscr{A})$ for each $\mathscr{A}\in X^*$ are clear.

ii) If $A\in f^*(\mathscr{A})$ and $A\subset B$, then

$$f^{-1}(w(A))\in\mathscr{A},\ f^{-1}(w(A))\subset f^{-1}(w(B))\in\mathscr{A}.$$

Thus $B\in f^*(\mathscr{A})$.

iii) If $w(A)\in f^*(\mathscr{A})$, $f^{-1}(w(w(A)))\in\mathscr{A}\Leftrightarrow f^{-1}(w(A))\in\mathscr{A}\Leftrightarrow A\in f^*(\mathscr{A})$.

Thus $A\in f^*(\mathscr{A})$. Therefore, f is a function.

(2) Since $f(\mathscr{A})=\{f(A)\,|\,A\in\mathscr{A}\}$ and $f^*(\mathscr{A})=\{B\subset Y\,|\,f^{-1}(w(B))\in\mathscr{A}\}$, for each $B\in f(\mathscr{A})$, there exists $A\in\mathscr{A}$ such that $B=f(A)$. Thus $A\subset f^{-1}(B)\subset f^{-1}(w(B))\Rightarrow B\in f^*(\mathscr{A})$. Therefore, $f(\mathscr{A})\subset f^*(\mathscr{A})$ for each $\mathscr{A}\in X^*$.

(3) Let $A\in f^*(\mathscr{A}_u(x))$. Then $f^{-1}(w(A))\in\mathscr{A}_u(x)\Leftrightarrow x\in u(f^{-1}(w(A)))\Leftrightarrow f(x)\in f(u(f^{-1}(w(A))))=w(f(f^{-1}(w(A)))\Leftrightarrow f(x)\in w(w(A))=w(A)\Leftrightarrow A\in\mathscr{A}_w(f(x))$. Therefore, $f^*(\mathscr{A}_u(x))=\mathscr{A}_w(f(x))$ for each $x\in X$.

$$\begin{array}{ccc}(X, u) & \xrightarrow{\quad f \quad} & (Y, w)\\ \varphi_X\big\downarrow & \circlearrowright & \big\downarrow\varphi_Y\\ (X^*, k_u) & \xrightarrow{\quad f^* \quad} & (Y^*, k_w)\end{array}$$

By the above diagram, we now have

$$(f^*\circ\varphi_X)(x)=f^*(\mathscr{A}_u(x))=\mathscr{A}_w(f(x))=(\varphi_Y\circ f)(x),$$

for each $x\in X$.

(4) Let $\alpha\subset X^*$ and let $\mathscr{A}\in k_u(\alpha)=\{\mathscr{A}\in X^*\,|\,\cap\alpha\subset\mathscr{A}\}$. Suppose that $f^*(\mathscr{A})\notin k_w(f^*(\alpha))=\{\mathfrak{B}\in Y^*\,|$

$\cap f^*(\alpha)\subset\mathfrak{B}\}$. Then $\cap f^*(\alpha)\subset f^*(\mathscr{A})$ and so for some $A\in\cap f^*(\alpha)$ $A\notin f^*(\mathscr{A})$ (that is, $f^{-1}(w(A))$ $\notin\mathscr{A}$). Since $\cap\alpha\subset\mathscr{A}$, there exists an $\zeta\in\alpha$ such that $f^{-1}(w(A))\notin\zeta$ (*i.e.*, $A\notin f^*(\zeta)$). This contradicts $A\in f^*(\zeta)$.

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