On Super-continuous Functions

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1. Introduction

In 1980, T. Noiri [8] has introduced the concept of strongly \( \theta \)-continuous functions which has been investigated by P.E. Long and L.L. Herrington [4]. Quite recently, B.M. Munshi and D.S. Bassan [6] has introduced a new class of functions, called super-continuous functions, which contains the class of strongly \( \theta \)-continuous functions and is contained in the class of continuous functions.

The purpose of the present note is to investigate some properties of super-continuity for product spaces and the relationships between super-continuity and functions with \( \delta \)-closed graphs due to T. Noiri [8].

2. Preliminaries

Throughout this paper, spaces mean always topological spaces. Let \( S \) be a subset of a space \( X \). The closure of \( S \) and the interior of \( S \) are denoted by \( Cl(S) \) and \( Int(S) \), respectively. A point \( x \in X \) is said to be \( \delta \)-cluster point of \( S \) if \( S \cap Int(Cl(U)) \neq \emptyset \) for each open set \( U \) containing \( x \). The set of all \( \delta \)-cluster points of \( S \) is called the \( \delta \)-closure of \( S \) denoted by \( Cl_\delta(S) \). If \( Cl_\delta(S) = S \), then \( S \) is called \( \delta \)-closed. The complement of a \( \delta \)-closed set is called \( \delta \)-open.

Definition 2.1. A function \( f : X \to Y \) is said to be super-continuous [6] (resp. \( \delta \)-continuous [8] and \( \theta \)-continuous [2]) if for each \( x \in X \) and each open neighborhood \( V \) of \( f(x) \), there exists an open neighborhood \( U \) of \( x \) such that \( f(Int(Cl(U))) \subseteq V \) (resp. \( f(Int(Cl(U))) \subseteq Int(Cl(V)) \) and \( f(Cl(U)) \subseteq Cl(V) \)).

It is obvious that super-continuity implies \( \delta \)-continuity. However, the converse is not true, as the following examples show:

Example 2.2. Let \( X = \{ a, b, c, d \} \) and \( \tau = \{ X, \phi, \{ c \}, \{ a, b \}, \{ a, b, c \} \} \). Let \( Y = \{ p, q, r, w \} \) and \( \sigma = \{ Y, \phi, \{ p \}, \{ r \}, \{ p, q \}, \{ p, r \}, \{ p, q, r \}, \{ p, r, w \} \} \). Define a function \( f : (X, \tau) \to (Y, \sigma) \) as follows:
\[
f(a) = f(b) = q \quad \text{and} \quad f(c) = f(d) = p.
\]
Then \( f \) is \( \delta \)-continuous, but it is not super-continuous.

3. Product spaces

Theorem 3.1. If \( f : X \to Y \) is super-continuous and \( g : Y \to Z \) is continuous, then \( g \circ f : X \to Z \) is super-continuous.

Theorem 3.2. Let \( f : X \to \Pi X \) be a super-continuous function, then \( P_a \circ f \) is super-continuous for each
\( \alpha \in \mathcal{P} \), where \( P_\alpha \) is the projection \( II X_\alpha \) onto \( X_\alpha \).

**Proof.** This is an immediate consequence of Theorem 3.1.

**Theorem 3.3.** Let \( f : X \to Y \) be a function and \( g : X \to X \times Y \) the graph function of \( f \) defined by \( g(x) = (x, f(x)) \) for each \( x \in X \). Then \( g \) is super-continuous if and only if \( f \) is super-continuous and \( X \) is semi-regular.

**Proof.** Suppose that \( g \) is super-continuous. By Theorem 3.2, \( f = P_\alpha g \) is super-continuous and the identity \( P_\alpha g \) is also super-continuous. Thus, for each open set \( V \) in \( X \) and each \( \alpha \in \mathcal{P} \), by Theorem 2.1 of [6], there exists an open set \( U \) in \( X \) such that \( x \in U \subset \text{Int}(\text{Cl}(U)) \subset V \). This shows that \( X \) is semi-regular.

Conversely, suppose that \( f \) is super-continuous. Let \( x \in X \) and \( W \) be an open set in \( X \times Y \) containing \( g(x) \). Then there exist open sets \( G \) and \( V \) in \( X \) and \( Y \), respectively, such that \( g(x) = (x, f(x)) \in G \times V \subset W \). Since \( X \) is semi-regular, there exists an open set \( U \) in \( X \) containing \( x \) such that \( \text{Int}(\text{Cl}(U)) \subset G \) and \( f(\text{Int}(\text{Cl}(U))) \subset V \). Therefore, we have \( g(\text{Int}(\text{Cl}(U))) \subset G \times V \subset W \). This shows that \( g \) is super-continuous.

**Theorem 3.4.** Let \( f_\alpha : X_\alpha \to Y_\alpha \) be a function for each \( \alpha \in \mathcal{P} \) and \( f : II X_\alpha \to II Y_\alpha \) a function defined by \( f([x_\alpha]) = [f(x_\alpha)] \) for each \( [x_\alpha] \in II X_\alpha \). Then \( f \) is super-continuous if and only if \( f_\alpha \) is super-continuous for each \( \alpha \in \mathcal{P} \).

**Proof.** Suppose that \( f \) is super-continuous. Let \( V_\alpha \) be an open set in \( Y_\alpha \). Then \( V = V_\alpha \times II \{ Y_\beta | \beta \in \mathcal{P} - \{ \alpha \} \} \) is open in \( II Y_\alpha \). By [6], \( f^{-1}(V) = f_\alpha^{-1}(V_\alpha) \times II \{ X_\beta | \beta \in \mathcal{P} - \{ \alpha \} \} \) is \( \delta \)-open in \( II X_\alpha \). It follows easily from [1] and Theorem 6 of [5] that \( f_\alpha^{-1}(V_\alpha) \) is \( \delta \)-open in \( X_\alpha \). Thus \( f_\alpha \) is super-continuous.

Conversely, suppose that \( f_\alpha \) is super-continuous for each \( \alpha \in \mathcal{P} \). Let \( V = V_\alpha \times II \{ Y_\beta | \beta \in \mathcal{P} - \{ \alpha \} \} \) be an open set in \( II Y_\alpha \). Then by [1] and Theorem 5 of [5], \( f^{-1}(V) = f^{-1}(V_\alpha) \times II \{ X_\beta | \beta \in \mathcal{P} - \{ \alpha \} \} \) is \( \delta \)-open in \( II X_\alpha \) since \( f_\alpha^{-1}(V_\alpha) \) is \( \delta \)-open in \( X_\alpha \). This shows that \( f \) is super-continuous.

4. Functions with \( \delta \)-closed graphs

For a function \( f : X \to Y \), the subset \( \{(x, f(x)) | x \in X\} \) of the product space \( X \times Y \) is called the **graph** of \( f \) and denoted by \( G(f) \).

**Definition 4.1.** The graph \( G(f) \) of a function \( f : X \to Y \) is said to be \( \delta \)-closed [8] if \( G(f) \) is \( \delta \)-closed in \( X \times Y \).

**Theorem 4.2.** If \( f : X \to Y \) is \( \theta \)-continuous and \( Y \) is Hausdorff, then \( G(f) \) is \( \delta \)-closed in \( X \times Y \).

**Proof.** Let \( (x,y) \in G(f) \). Then \( y \neq f(x) \) and there are open sets \( V \) and \( W \) containing \( f(x) \) and \( y \), respectively, such that \( \text{Int}(\text{Cl}(V)) \cap \text{Int}(\text{Cl}(W)) = \phi \) by Theorem 6 of [7]. Therefore we obtain \( \text{Cl}(V) \cap \text{Int}(\text{Cl}(W)) = \phi \). By \( \theta \)-continuity of \( f \), there exists an open set \( U \) containing \( x \) such that \( f(\text{Cl}(U)) \cap \text{Int}(\text{Cl}(W)) = \phi \). By Lemma of [3], \( [\text{Int}(\text{Cl}(U)) \times \text{Int}(\text{Cl}(W))] \cap G(f) = \phi \). This shows that \( G(f) \) is \( \delta \)-closed.

The following Corollary 4.3 follows immediately from Theorem 4.2.

**Corollary 4.3.** (T. Noiri [8]). If \( f : X \to Y \) is \( \delta \)-continuous and \( Y \) is Hausdorff, then \( G(f) \) is
δ-closed in $X \times Y$.

**Theorem 4.4.** Let $f : X \rightarrow Y$ be a function with a δ-closed graph. If $K$ is compact in $Y$ (resp. $X$), then $f^{-1}(K)$ (resp. $f(K)$) is δ-closed in $X$ (resp. $Y$).

**Proof.** We prove only the first case, the proof of the second being analogous. Suppose that $K$ is compact in $Y$. For each $x \in f^{-1}(K)$ and $y \in K$, we have $(x, y) \in G(f)$. Hence there exist open sets $U_x(x)$ and $W(y)$ containing $x$ and $y$, respectively, such that $\cap \text{Int} \cap (\text{Cl}(U_x(x)) \times \text{Int} \cap (\text{Cl}(W(y))))) \cap G(f) = \phi$. By [3], we obtain $\cap \text{Int}(\text{Cl}(U_x(x))) \cap \text{Int}(\text{Cl}(W(y))) = \phi$. Thus, there exists a finite subcover $\{W(x)\}_{x \in A}$, where $A$ is a finite subset of $K$. Moreover, the corresponding $U_y(x)$ have the property that $\cap \text{Int}(\text{Cl}(U_y(x))) \cap \text{Int}(\text{Cl}(W(y))) = \phi$. Consider $U = \bigcup_{a \in A} Y_a(x)$. Then $U$ is an open set containing $x$ and we obtain $\cap \text{Int}(\text{Cl}(U)) \cap K \cap f^{-1}(\cup \text{Int}(\text{Cl}(U_y(x))) \cap \text{Int}(\text{Cl}(W(x))) = \phi$. Therefore, $\cap \text{Int}(\text{Cl}(U)) \cap K = \phi$. Thus, we have $\text{Int}(\text{Cl}(U)) \cap f^{-1}(K) = \phi$ and hence $x \in \text{Cl}_f(f^{-1}(K))$. This shows that $f^{-1}(K)$ is δ-closed in $X$.

**Theorem 4.5.** Let $f : X \rightarrow Y$ be a function with a δ-closed graph. If $Y$ is compact, then $f$ is super-continuous.

**Proof.** Let $F$ be any closed set of $Y$. By Theorem 4.4, $f^{-1}(F)$ is δ-closed in $X$. By Theorem 2. 1 of [6], $f$ is super-continuous.

**Theorem 4.6.** Let $f : X \rightarrow Y$ be $\theta$-continuous and $Y$ be Hausdorff.

i) If $Y$ is compact, then $f$ is super continuous.

ii) If $Y$ is nearly-compact, then $f$ is δ-continuous.

**Proof.** i) By Theorem 4.2 and 4.5, it is obvious.

ii) By Theorem 4.2 and Theorem 5.2 of [8], it is obvious.

**References**