韓國數學教育學會誌<數學教育>
1984. 3. Vol. XXII, No. 2.

Journal of the Korea Society of Mathematical Education
Mar. 1984. Vol. XXII, No. 2.

On Super-continuous Functions

Bae Hun Park and Sin Min Kang
Gyeongsang National University, Jinju, Korea

1. Introduction

In 1980, T. Noiri [8] has introduced the concept of strongly θ-contiuous functions which has been investigated by P.E. Long and L.L. Herrington [4]. Quite recently, B.M. Munshi and D.S. Bassan [6] has introduced a new class of functions, called super-continuous functions, which contains the class of srtongly θ-continuous functions and is contained in the class of continuous functions.

The purpose of the present note is to investigate some properties of super-continuity for product spaces and the relationships between super-continuity and functions with δ-closed graphs due to T. Noiri [8].

2. Preliminaries

Throughout this paper, spaces mean always topological spaces. Let S be a subset of a space X. The closure of S and the interior of S are denoted by $Cl(S)$ and $Int(S)$, respectively. A point $x\in X$ is said to be *δ-cluster* point of S[9] if $S\cap Int(Cl(U))\neq\phi$ for each open set U containing x. The set of all δ-cluster points of S is called the *δ-closure* of S and denoted by $Cl_\delta(S)$. If $Cl_\delta(S)=S$, then S is called *δ-closed*. The complement of a δ-closed set is called *δ-open*.

Definition 2.1. A function $f: X\to Y$ is said to be *super-continuous* [6] (resp. *δ-continuous* [8] and *θ-continuous* [2]) if for each $x\in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(\mathrm{Int}(Cl(U)))\subset V$(resp. $f(\mathrm{Int}(Cl(U)))\subset\mathrm{Int}(Cl(V))$ and $f(Cl(U))\subset Cl(V)$.

It is obvious that super-continuity implies δ-continuity. However, the converse is not true, as the following examples show:

Example 2.2. Let $X=\{a,b,c,d\}$ and $\tau=\{X,\phi,\{c\},\{a,b\},\{a,b,c\}\}$. Let $Y=\{p,q,r,w\}$ and $\sigma=\{Y,\phi,\{p\},\{r\},\{p,q\},\{p,r\},\{p,q,r\},\{p,r,w\}\}$. Define a function $f:(X,\tau)\to(Y,\sigma)$ as follows: $f(a)=f(b)=q$ and $f(c)=f(d)=p$.

Then f is δ-continuous, but it is not super-continuous.

3. Product spaces

Theorem 3.1. *If $f: X\to Y$ is super-continuous and $g: Y\to Z$ is continuous, then $g\circ f: X\to Z$ is super-continuous.*

Theorem 3.2. *Let $f: X\to\Pi X_\alpha$ be a super-continuous function, then $P_\alpha\circ f$ is super-continuous for each*

$\alpha \in \nabla$, *where* P_α *is the projection* ΠX_α *onto* X_α.

Proof. This is an immediate consequence of Theorem 3.1.

Theorem 3.3. *Let* $f: X \to Y$ *be a function and* $g: X \to X \times Y$ *the graph function of* f *defined by* $g(x)=(x, f(x))$ *for each* $x \in X$. *Then* g *is super-continuous if and only if* f *is super-continuous and* X *is semi-regular.*

Proof. Suppose that g is super-continuous. By Theorem 3.2, $f=P_y \circ g$ is super-continuous and the identity $P_x \circ g$ is also super-continuous. Thus, for each open set V in X and each $x \in V$, by Theorem 2. 1 of [6], there exists an open set U in X such that $x \in U \subset Int(Cl(U)) \subset V$. This shows that X is semi-regular.

Conversely, suppose that f is super-continuous. Let $x \in X$ and W be an open set in $X \times Y$ containing $g(x)$. Then there exist open sets G and V in X and Y, respectively, such that $g(x)=(x, f(x)) \in G \times V \subset W$. Since X is semi-regular, there exists an open set U in X containing x such that $Int(Cl(U)) \subset G$ and $f(Int(Cl(U))) \subset V$. Therefore, we have $g(Int(Cl(U))) \subset G \times V \subset W$. This shows that g is super-continuous.

Theorem 3.4. *Let* $f_\alpha: X_\alpha \to Y_\alpha$ *be a function for each* $\alpha \in \nabla$ *and* $f: \Pi X_\alpha \to \Pi Y_\alpha$ *a function defined by* $f(\{x_\alpha\})=\{f(x_\alpha)\}$ *for each* $\{x_\alpha\} \in \Pi X_\alpha$. *Then* f *is super-continuous if and only if* f_α *is super-continuous for each* $\alpha \in \nabla$.

Proof. Suppose that f is super-continuous. Let V_α be an open set in Y_α. Then $V=V_\alpha \times \Pi\{Y_\beta | \beta \in \nabla - \{\alpha\}\}$ is open in ΠY_α. By [6], $f^{-1}(V)=f_\alpha^{-1}(V_\alpha) \times \Pi\{X_\beta | \beta \in \nabla - \{\alpha\}\}$ is δ-open in ΠX_α. It follows easily from [1] and Theorem 6 of [5] that $f_\alpha^{-1}(V_\alpha)$ is δ-open in X_α. Thus f_α is supercontinuous.

Conversely, suppose that f_α is super-continuous for each $\alpha \in \nabla$. Let $V=V_\alpha \times \Pi\{Y_\beta | \beta \in \nabla - \{\alpha\}\}$ be an open set in ΠY_α. Then by [1] and Theorem 5 of [5], $f^{-1}(V)=f^{-1}(V_\alpha) \times \Pi\{X_\beta | \beta \in \nabla - \{a\}\}$ is δ-open in ΠX_α since $f_\alpha^{-1}(V_\alpha)$ is δ-open in X_α. This shows that f is super-continuous.

4. Functions with δ-closed graphs

For a function $f: X \to Y$, the subset $\{(x, f(x)) | x \in X\}$ of the product space $X \times Y$ is called the *graph* of f and denoted by $G(f)$.

Definition 4.1. The graph $G(f)$ of a function $f: X \to Y$ is said to be δ-closed [8] if $G(f)$ is *δ-closed* in $X \times Y$.

Theorem 4.2. *If* $f: X \to Y$ *is* θ*-continuous and* Y *is Hausdorff, then* $G(f)$ *is* δ*-closed in* $X \times Y$.

Proof. Let $(x, y) \notin G(f)$, Then $y \neq f(x)$ and there are open sets V and W containing $f(x)$ and y, respectively, such that $Int(Cl(V)) \cap Int(Cl(W))=\phi$ by Theorem 6 of [7]. Therefore we obtain $Cl(V) \cap Int(Cl(W))=\phi$. By θ-continuity of f, there exists an open set U containing x such that $f(Cl(U)) \cap Int(Cl(W))=\phi$. By Lemma of [3], $[Int(Cl(U)) \times Int(Cl(W))] \cap G(f)=\phi$. This shows that $G(f)$ is δ-closed.

The following Corollary 4.3 follows immediately from Theorem 4.2.

Corollary 4.3. (T. Noiri [8]). *If* $f: X \to Y$ *is* δ*-continuous and* Y *is Hausdorff, then* $G(f)$ *is*

δ-closed in $X\times Y$.

Theorem 4.4. *Let $f: X\to Y$ be a function with a δ-closed graph. If K is compact in Y (resp. X), then $f^{-1}(K)$ (resp. $f(K)$) is δ-closed in X (resp. Y).*

Proof. We prove only the first case, the proof of the second being analogous. Suppose that K is compact in Y. For each $x\notin f^{-1}(K)$ and $y\in K$, we have $(x,y)\notin G(f)$. Hence there exist open sets $U_y(x)$ and $W(y)$ containing x and y, respectively, such that $[Int(Cl(U_y(x)))\times Int(Cl(W(y)))]\cap G(f)=\phi$. By [3], we obtain $f(Int(Cl(U_y(x))))\cap Int(Cl(W(y)))=\phi$. Thus, there exists a finite subcover $\{W(y_\alpha)\mid\alpha\in\Delta$, where Δ is a finite$\}$ of K. Moreover, the corresponding $Uy_\alpha(x)$ have the property that $f(\bigcap_{\alpha\in\Delta} Int(Cl(Uy_\alpha(x)))\cap[\bigcup_{\alpha\in\Delta} Int(Cl(W(y_\alpha)))]=\phi$. Consider $U=\bigcap_{\alpha\in\Delta} Uy_\alpha(x)$. Then U is an open set containing x and we obtain $f(Int(Cl(U)))\cap K\subset f(\bigcap_{\alpha\in\Delta} Int(Cl(Uy_\alpha(x)))\cap[\bigcup_{\alpha\in\Delta} W(y_\alpha)]=\phi$. Therefore, $f(Int(Cl(U)))\cap K=\phi$. Thus, we have $Int(Cl(U))\cap f^{-1}(K)=\phi$ and hence $x\notin Cl_\delta(f^{-1}(K))$. This shows that $f^{-1}(K)$ is δ-closed in X.

Theorem 4.5. *Let $f: X\to Y$ be a function with a δ-closed graph. If Y is compact, then f is super-continuous.*

Proof. Let F be any closed set of Y. By Theorem 4.4, $f^{-1}(F)$ is δ-closed in X. By Theorem 2.1 of [6], f is super-continuous.

Theorem 4.6. *Let $f: X\to Y$ be θ-continuous and Y be Hausdorff.*

i) If Y is compact, then f is super continuous.

ii) If Y is nearly-compact, then f is δ-continuous.

Proof. i) By Theorem 4.2 and 4.5, it is obvious.

ii) By Theorem 4.2 and Theorem 5.2 of [8], it is obvious.

References

1. J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1968.
2. S.V Formin, Extensions of topological spaces, *Dokl. Akud. Nauk SSSR*, 32(1941), 114-116=*Ann. of Math.*, 44(1943), 471-480.
3. P.E. Long, Functions with closed graphs, *Amer. Math. Monthly*, (8)76(1969), 930-932.
4. P.E. Long and L.L. Herrington, Strongly θ-continuous functions, *J. Korean Math. Soc.*, 18(1981), 21-28.
5. P.E. Long and L.L., The T_θ-topology and faintly continuous functions, *Kyungpook Math. J.*, 22(1982), 7-14.
6. B.M. Munshi and D.S. Bassan, Super-continuous mappings, *Indian J. Pure and Applied Math.*, 13(1982), 229-236.
7. T. Noiri, Between continuity and weak continuity, *Bollettino Un. Mate. Ital.*, (4)9(1974), 647-654.
8. T. Noiri, On δ-continuous functions, *J. Korean Math. Soc.*, 16 (1980), 161-166.
9. N.V. Veličko, H-closed topological spaces, *Amer. Math. Soc. Transl.*, (2)78(1968), 103-118.