Near-rings with IFP

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G. Mason showed the following theorem in [2].

Theorem. [proposition 1 [2]]. If a zero symmetric near-ring $N$ is unital then the left regularity, the right regularity and the left strongly regularity are equivalent each other.

In this paper we generalize his theorem to non-zero symmetric near-ring partially.

A near-ring $N$ is a system $(N, +, \cdot)$ such that $(N, +)$ is a group, $(N, \cdot)$ a monoid and the right distributive law holds. $N$ is regular if for all $x$ in $N$, there exists $a$ in $N$ with $x = xax$, and $N$ is right (left) strongly regular if for all $x$ in $N$, there exists $a$ in $N$ with $x = x^2a(x = ax^2)$ [2]. $N$ is called a right (left) regular if $N$ is regular and right (left) strongly regular [2]. Undefined terminology refer to [1].

Definition 1 [2]. A near-ring $N$ is with IFP if for some $a, b$ in $N$, $ab = 0$ implies $axb = 0$ for each $x$ in $N$.

Theorem 2. Let $N$ be a right strongly regular near-ring with IFP. Then $N$ is right regular.

Proof. Assume that $x = x^2a$. It implies that $x^2 = x^2ax$. Since $N$ is with IFP, $(x - x^2a)ax = 0$, so $xax = x^2a^2x$. Let $b = xa^2$, then $xb = x^2a^2x = xax = x$ and $x^2b = x^2a^2 = x^2a = x$. Thus $N$ is right regular.

Remark If $N$ is a left (or right) strongly regular near-ring, then it is reduced [2]. If $N$ is a zero symmetric reduced near-ring, then it is with IFP [2].

Lemma 3. [2]. Let $N$ be a left regular with IFP. Then $N$ is right regular.

Theorem 4. Let $N$ be an unital near-ring with IFP. Then the left regularity is equivalent to right regularity.

Proof. Assume that $x = x^2a = xax$ for some $a$ in $N$. It implies that $0 = x - xax = (1 - xa)x$ where $1$ is the identity in $N$. Since $N$ is with IFP, $ax - xa^2x = 0$. Thus $ax = xa^2$. Since $x^2a^2x - x = x^2a^2x - xax = (x^2a - x)ax = 0$, $x = x^2a^2x$. It follows that $x^2 = (xa^2x)(xa^2x) = xa^2x = ax$. Thus $x = xa^2x = ax$. Thus $x = xa^2 = x$. Hence $N$ is a left regular near-ring.

By lemma, the converse is true.

Lemma 5. Let $N$ be an unital near-ring with IFP. Then the regularity is equivalent the right regularity.

Proof. Assume that $x = xax$ for some $a$ in $N$. It follows that $(1 - xa)x = 0$. Since $N$ is with IFP, $(1 - xa)ax = 0$ so $ax = xa^2x$. Put $b = a^2x$ then $x^2b = x^2(a^2x) = xax = x$. Hence $N$ is right regular.

The converse is true, in general.
Theorem 6. Let $N$ be an unital near-ring with IFP. Then the followings are equivalent.

1) $N$ is regular.
2) $N$ is right regular.
3) $N$ is right strongly regular.
4) $N$ is left regular.

Remark. If $N$ is zero symmetric, these are equivalent to left strongly regular [2].

Reference