On the Conditions for One-sided Inverses to be Two-sided in Group Algebras

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1. Introduction. This paper contains some remarks about the Kaplansky's conjecture [1]: In a group algebra, are one-sided inverses two-sided? Hilbert space methods in the discrete group algebra have proved this to be true in characteristic 0; see [3]. But the characteristic p case remains open.

G. Losey showed this property (that one-sided inverses are two-sided) holds for group algebras of particular groups over arbitrary fields such that finite groups, abelian groups, nilpotent groups, and free groups [2].

Here we examine whether or not the conjecture is true in group algebras of supersolvable groups and solvable groups.

2. Preliminaries. A group $G$ is supersolvable if it has a normal series with cyclic factors:
\[ G = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq \ldots \supseteq G_\ast = \{1\}. \]

Theorem 1. [4] Let $RG$ be a group ring of a supersolvable group $G$ over a commutative ring with identity $R$. Suppose that $R$ has no nontrivial idempotents and that if $G$ has an element of prime order $p$ then $p \notin UR$. Then $RG$ has no nontrivial idempotents.

A group $G$ is an $F$-group if $RG$ is strongly finite for all strongly finite rings $R$ (A ring $R$ is strongly finite means that for any pair $A, B$ of $n \times n$ matrices over $R$, $AB = I_n$ implies $BA = I_n$ for all positive integers $n$). We know that $G$ is an $F$-group if and only if $RG$ is 1-finite, i.e., each right invertible element is left invertible, since $(RG)_n = (R_n)G$.

Lemma 1. A group $G$ is an $F$-group if and only if it is locally an $F$-group.

Theorem 2. [2] Let $G$ be a group and $H$ a subgroup of $G$ of finite index. If $H$ is an $F$-group, then $G$ is an $F$-group.

3. Some Results. First we give a result for supersolvable groups:

Theorem 3. Let $G$ be a supersolvable group and $K$ be a field of arbitrary characteristic. Then each element which is right-invertible is left-invertible in the group algebra $NG$.

Proof. Let $ab = 1$ for $a, b \in KG$. We set $e = ba$. Then $e$ is an idempotent, since $e^2 = (ba)(ba) = b(ab)a = ba = e$. Hence $e$ is trivial by Theorem 1, i.e., $e = 0$ or 1. But if $e = 0$, then $1 = (ab)(ab) = a(ba)b = 0$, a contradiction. Therefore, $ba = 1$.

Remark. In fact, this theorem remains true for a group ring over a commutative ring $R$ with
identity. ([2], Theorem 1)

**Theorem 4.** A solvable torsion group $G$ is an $\mathcal{F}$-group.

**Proof.** There exists a normal sequence $G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_n = [1]$, where $G_i/G_{i+1}$ is abelian for each $i$. Since $g$ is a torsion group, $G_i/G_{i+1}$ is an abelian torsion group, so it is locally finite for all $i$. Hence $G$ is also locally finite. By Lemma 1 and Theorem 2, $G$ is an $\mathcal{F}$-group.

Now we investigate our assertion for solvable groups. But in this direction, there are two short theorems:

**Theorem 5.** ([2]) Let $G$ be a group, $N$ a normal subgroup and assume (a) $G/N$ is abelian, (b) $N$ is finite. Then $G$ is an $\mathcal{F}$-group.

**Proof.** It is sufficient to assume $G/N$ finitely generated. Then $G/N \cong G_1/N \times \ldots \times G_s/N$, where each factor $G_i/N$ is cyclic. If each $G_i/N$ is finite cyclic, then $G_i/N$ is an $\mathcal{F}$-group by Theorem 2, and hence so is $G/N$. For the case that $G_i/N$ is infinite cyclic, let $xN$ be a generator of $N$. Then $y \rightarrow x^{-1}yx$ is an automorphism of $N$. Since $N$ is finite, $N$ has a finite automorphism group, so $x^m$ centralizes $N$ for some $m > 0$. Thus $N^* = \langle x^m, N \rangle = \langle x^m \rangle \times N$ is an $\mathcal{F}$-group and $[G:N] = m$. By Theorem 2, $G$ is an $\mathcal{F}$-group.

**Theorem 6.** ([2]) Let $G$ be a group, $N$ a normal subgroup and assume (a) $G/N$ is abelian, (b) $N$ is finitely generated abelian. Then $G$ is an $\mathcal{F}$-group.

From this, we obtain the following corollaries:

Let $C(G)$ be the center of a group $G$.

**Corollary 1.** If $C(G)$ has a finite index in $G$, then $G$ is an $\mathcal{F}$-group.

**Proof.** By Schur's lemma [4], the commutator subgroup $G'$ is finite. And $G/G'$ is abelian. So the result follows by Theorem 5.

A group in which each element has a finite number of conjugates is called an $FC$-group.

**Corollary 2.** A finitely generated $FC$-group is an $\mathcal{F}$-group.

**Proof.** Let $g_1, g_2, \ldots, g_s$ be generators of $G$. Then $[G:C(G)] < \infty$ for all $i$ and consequently $[G:C(G)] = [G : \langle g_i \rangle \cap C(G)] < \infty$. Thus $G$ is an $\mathcal{F}$-group by Corollary 1.

**References**