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An Application of Regular Group Forms

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1, Introduction

It is a well-known fact that if q is an isotropic quadratic form, then q is universal. Furthermore, any quadratic form q with $dim\ q \geqq 5$ over p-adic field Q_p $(p \geqq 3)$ is always isotropic (5; p. 36). But, in general, any universal quadratic form is not always isotropic. Note that every regular quadratic form of dimension$\geqq 2$ over a finite field is always universal.

In this paper, we shall prove that if q is a universal regular quadratic form with $dim\ q=3$ or 5, then it is isotropic.

2. Preliminaries

Throughout this paper, a field shall always mean a field of characteristic not equal to 2. A *quadratic form* q over a field F, by definition, a homogeneous polynomial q over F of degree 2:

$$q(X)=q(x_1, \dots, x_n)=\sum a_{ij}x_ix_j,\ a_{ij}\in F.$$

Since we assume $charF \neq 2$, we may symmetrize the coefficients to assume $a_{ij}=a_{ji}$ for all i, j.

Let F^n denote the space of n-tuples. Any quadratic form q gives rise to a map $q: F^n \to F$. We shall refer to $q: F^n \to F$ as the *quadratic map* defined by the quadratic form q. Since the quadratic map determines uniquely the quadratic form q, we can regard the quadratic maps as the quadratic forms.

It q and q' are quadratic forms, we say that they are *isometric* $(\simeq)$ if there exists a linear automorphism $g: F^n \to F^n$ such that

$$q'(gx)=q(x) \qquad \text{for all } x \text{ in } F^n.$$

Let $q(x)=\sum a_{ij}x_ix_j$ be a quadratic form. We shall say that q is *regular* if the symmetric matrix (a_{ij}) is invertible. In this case, $d(q)=det\ (a_{ij})\dot{F}^2$ (an element of $\dot{F}/\dot{F}^2$) is called the *determinant* of the regular quadratic form q.

Definition 2.1. Let q be a quadratic form. Let $e_1, \dots, e_n$ form a basis of F^n. we shall say that $e_1, \dots, e_n$ forms an *orthogonal basis* for q if $q(e_i+e_j)=q(e_i)+q(e_j)$ for all $i \neq j$.

It is clear that any orthogonal linearly independent vectors of q can be extended to an orthogonal basis of q. In other words, any quradatic form q is isometric to some *diagonal form*, $d_1x_1{}^2+\dots+d_nx_n{}^2$.

Definition 2.2. Let $q_1: F^n \to F$, and $q_2: F^m \to F$ be quadratic forms. We shall say that a quadratic form q is an *orthogonal sum* of q_1 and q_2 $(q=q_1 \perp q_2)$ if $q(x_1+x_2)=q_1(x_1)+q_2(x_2)$ for all (x_1, x_2) in $F^n \times F^m$. We shall say that a quadratic form q is a tensor product of q_1 and q_2 $(q=q_1 \otimes q_2= q_1q_2)$, if $q(x_1 \otimes x_2)=q_1(x_1)q_2(x_2)$ for all $x_1 \otimes x_2$ in $F^n \otimes F^m$.

In the sequel, we shall abbreviate $d_1x_1^2+\ldots+d_nx_n^2$ by $\langle d_1, \ldots, d_n\rangle$. The special quadratic form $q\perp\ldots\perp q$ will be abbreviated as nq.

Definition 2.3. Let q be a quadratic form over F. We say that a non-zero vector x in F^n is *isotropic* if $q(x)=0$, and say that x is *anisotropic* if otherwise. The quadratic form is said be *isotropic* if it contains a non-zero isotropic vector, and is said to be *anisotropic* if otherwise.

Theorem 2.1. *Let q be a quadratic form with dim $q=2$. The following four statements are equivalent:*

(1) *q is regular and isotropic.*

(2) $d(q)=-1\dot{F}^2$.

(3) $q\simeq\langle 1, -1\rangle$.

(4) $q\simeq xy$.

Proof. See (2; p.12) Q.E.D.

The 2-dimensional quadratic form satisfying the conditions in theorem 2.1 is called the ***hyperbolic plane*** and will be denoted by H. An orthogonal sum of hyperbolic planes will be called a ***hyperbolic space.***

Corollary 2.2. *If q is any regular quadratic form, then $q\otimes H\simeq(\dim q)H$.*

Proof. Inducting on *dim q*, we are reduced to the case when $q\simeq\langle a\rangle$, $a\neq 0$. But then, $\langle a\rangle\otimes H\simeq\langle a, -a\rangle\simeq H$ by theorem 2.1. Q.E.D.

Definition 2.4. Let q be a regluar quadratic form over a field F. $D(q)=\{a\in\dot{F}\,|\,a=q(x)$ for some x in $F^n\}$, the set of elements in $\dot{F}$ *represented* by q. In particular, q is called ***universal*** if $D(q)=\dot{F}$.

Theorem 2.3. *Let q be a regular quadratic form. Then*

(1) *q is isotropic iff $q\simeq H\perp f$ for some quadratic form f.*

(2) *If q is isotropic then it is universal.*

Proof. See (2; p.13). Q.E.D.

Theorem 2.4. *Let q be a quadratic form. For $a\in\dot{F}$, we have $a\in D(q)$ iff $q\simeq\langle a, a_2, \ldots, a_n\rangle$ for suitable $a_i\in\dot{F}$.*

Proof. "If" is clear. Conversely, take $a=q(u)$, $u\in F^n$. If we complete u to an orthogonal basis $u, u_2, \ldots, u_n$ of F^n, then $q\simeq\langle q(u), q(u_2), \ldots, q\langle u_n\rangle\rangle$. Q.E.D.

Corollary 2.5. *Let $q_1=\langle a, b\rangle$, $q_2=\langle c, d\rangle$ be regular quadratic forms. Then $q_1\simeq q_2$ iff $d(q_1)=d(q_2)$ and $D(q_1)\cap D(q_2)\neq\phi$.*

Proof. It is clear from theorem2.4. Q.E.D.

3. Results of Pfister forms

Let us first make formal definitions.

Definition 3.1. Let q be a regular quadratic form over a field F. $G(q)=\{a\in\dot{F}\,|\,\langle a\rangle q\simeq q\}$, the group of *similarity factors* of q.

Definition 3.2. We shall say that a regular quadratic form q is a ***group form*** over F, if $D(q)$ is a subgroup of $\dot{F}$.

Since $D(q)$ is stable under multiplication by $\dot{F}$, in order that $D(q)$ is a subgroup of $\dot{F}$, it is enough that $D(q)$ is closed under multiplication.

In this section, we shall try to cover some of basic facts about Pfister forms. Let us recall that an *n-fold Pfister form* over F means a quadratic form of the shape ${}_{i=1}\otimes^{n}\ \langle 1, a_i\rangle$, $a_i\in\dot{F}$, and is abbreviated by the notation:

$$\langle\langle a_1, \ldots, a_n\rangle\rangle.$$

Note the following two special cases when we set $a_1=\pm1$:

$$\langle\langle 1, a_2, \ldots, a_n\rangle\rangle\simeq 2\langle\langle a_2, \ldots, a_n\rangle\rangle,$$
$$\langle\langle -1, a_2, \ldots, a_n\rangle\rangle\simeq 2^{n-1}H.$$

Main theorem for Pfister forms 3.1. *Let* $q=\langle\langle a_1, \ldots, a_n\rangle\rangle$. *Then: MT* 1. $D(q)=G(q)$; *in particular, q is a group form. MT* 2. *If q is isotropic, it must be hyperbolic.*

We start with a lemma.

Lemma 3.2. (1) $\langle\langle a, b\rangle\rangle\simeq\langle\langle a, by\rangle\rangle$ *if* $y\in D\langle\langle a\rangle\rangle$.

(2) $\langle\langle a, b\rangle\rangle\simeq\langle\langle z, ab\rangle\rangle$ *if* $z\in D\langle a, b\rangle$.

Proof. (1) $\langle\langle a, b\rangle\rangle\simeq\langle 1, a\rangle\perp\langle b\rangle\langle 1, a\rangle$

$\simeq\langle 1, a\rangle\perp\langle b\rangle\langle y, ay\rangle$ by corollary 2.5

$\simeq\langle\langle a, by\rangle\rangle$.

(2) $\langle\langle a, b\rangle\rangle\simeq\langle 1, ab\rangle\perp\langle a, b\rangle$

$\simeq\langle 1, ab\rangle\perp\langle z, abz\rangle$ by corollary 2.5

$\simeq\langle\langle z, ab\rangle\rangle$. Q.E.D.

Since every n-fold Pfister form q represents 1, we may write $q\simeq\langle 1\rangle\perp q'$ by theorem 2.4. Here q' is uniquely determinded up to isometry by Witt's cancellation theorem, and we shall call q' the *pure subform* of q.

Theorem 3.3. *Let* $q=\langle\langle a_1, \ldots, a_n\rangle\rangle$ *and* $b\in\dot{F}$. *Then* $b\in D(q')$ *iff* $q\simeq\langle\langle b, b_2, \ldots, b_n\rangle\rangle$ *for suitable* $b_i\in\dot{F}$.

Proof. "If" is trivial, so we start with $b\in D(q')$. We shall use induction on n (the "fold" of q). If $n=1$, we have $q'\simeq\langle a_1\rangle\simeq\langle b\rangle$, so the desired conclusion is trivial. In general, write

$$f=\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle\simeq\langle 1\rangle\perp f',$$

so $q\simeq f\langle 1, a_n\rangle\simeq f'\perp\langle a_n\rangle f$

$q'\simeq f'\perp\langle a_n\rangle f$ (by cancellation of $\langle 1\rangle$).

From the hypothesis $b\in D(q')$, we can express b as $x'+a_n y$ where $x'\in D(f')\cup\{0\}$ and $y\in D(f)\cup\{0\}$. We can further express $y=t^2+y'$, where $t\in F$, and $y'\in D(f')\cup\{0\}$. By the inductive hypothesis, we may write $f\simeq\langle\langle x', c_2, \ldots, c_{n-1}\rangle\rangle$ (*unless* $x'=0$) and $f\simeq\langle\langle y', d_2, \ldots, d_{n-1}\rangle\rangle$ (*unless* $y'=0$). We may assume that $y\neq0$, for otherwise, $x'=b$, so

$$q\simeq f\langle\langle a_n\rangle\rangle\simeq\langle\langle x', c_2, \ldots, c_{n-1}\rangle\rangle\ \langle\langle a_n\rangle\rangle$$
$$\simeq\langle\langle b, c_2, \ldots, c_{n-1},\ a_n\rangle\rangle,$$

establishing the theorem. We claim that $q\simeq\langle\langle a_1, \ldots, a_{n-1},\ ya_n\rangle\rangle$.

For this, we may assume that $y'\neq0$, for otherwise, $y=t^2$, so

$$q=\langle\langle a_1, \ldots, a_n\rangle\rangle$$
$$\simeq\langle\langle a_1, \ldots, t^2a_n\rangle\rangle$$

$$\simeq \langle\langle a_1, \ldots, a_{n-1}, \; ya_n \rangle\rangle,$$

establishing our claim. Under this assumption, we have

$$y = t^2 + y' \in D\langle\langle y \rangle\rangle.$$

Then

$$q \simeq f\langle\langle a_n \rangle\rangle \simeq \langle\langle y', d_2, \ldots, d_{n-1}, a_n \rangle\rangle$$
$$\simeq \langle\langle y', d_2, \ldots, d_{n-1}, a_n y \rangle\rangle \quad \text{by lemma (1)}$$
$$\simeq f\langle\langle a_n \rangle\rangle \simeq \langle\langle a_1, \ldots, a_{n-1}, \; a_n y \rangle\rangle,$$

establishing the claim. Finally, we may assume that $x' \neq 0$ (for otherwise, $a_n y$ is already our b). Under this assumption, we have

$$q \simeq f\langle\langle a_n y \rangle\rangle \simeq \langle\langle x', c_2, \ldots, c_{n-1}, \; a_n y \rangle\rangle$$
$$\simeq \langle\langle x' + a_n y, c_2, \ldots, c_{n-1}, \; x' y a_n \rangle\rangle. \quad \text{by lemma (2)}$$
$$\simeq \langle\langle b, c_2, \ldots, c_{n-1}, \; x' y a_n \rangle\rangle. \quad \text{Q.E.D.}$$

Now, we can prove the "Main theorem for Pfister forms".

Proof of MT 2. If q is isotropic, we can write

$$\langle 1 \rangle \perp q' \simeq q \simeq \langle 1, -1 \rangle \perp \ldots \qquad \text{by theorem 2.3}$$

and cancellation yields $-1 \in D(q')$. By theorem 3.3, we have

$$q \simeq \langle\langle -1, \ldots \rangle\rangle \simeq 2^{n-1} H. \qquad \text{Q.E.D.}$$

Proof of MT 1. We need only show $D(q) \subset G(q)$, since $1 \in D(q)$. If $a \in D(q)$, then

$$\langle\langle -a \rangle\rangle q \simeq q \perp \langle -a \rangle q \simeq q \perp \langle -a, \ldots \rangle$$
$$\simeq \langle a, \ldots \rangle \perp \langle -a, \ldots \rangle$$

is isotropic, so must be hyperbolic by MT2. Hence

$$\langle\langle -a \rangle\rangle q \simeq 2^n H \simeq \langle\langle -1 \rangle\rangle q,$$

so cancellation of q yields

$$\langle -a \rangle q \simeq \langle -1 \rangle q,$$

so, tensoring by $\langle -1 \rangle$, we have $q \simeq \langle a \rangle q$. Q.E.D.

4. Applications

Theorem 4.1. *If $q \simeq \langle a, b, c \rangle$ represents $-abc \in \dot{F}$, then q is isotropic. In particular, a 3-dimensional regular quadratic form is universal iff it is isotropic.*

Proof. Let $f = \langle a, b, c, abc \rangle$. We claim that f is a 2-fold Pfister form. Since $-abc \in D(q) = D\langle a, b, c \rangle$, it follows that $ax^2 + by^2 + cz^2 = -abc$ for some x, y, z in F. Hence f is isotropic. By comparing detrminants and by theorem 2.4, we see that $f \simeq \langle 1, d, e, de \rangle$ for some d, e in $\dot{F}$. Thus f is a Pfister form. By theorem 3.1,

$$f = \langle a, b, c, abc \rangle \simeq \langle a \rangle f \simeq \langle 1, ab, ac, bc \rangle \simeq 2H \simeq \langle 1, -1, \; 1, -1 \rangle.$$

Hence $\langle ab, ac, bc \rangle$ is isotropic, so $\langle ab, ac, bc \rangle$ is a group form. Thus

$$q \simeq \langle abc \rangle \langle ab, ac, bc \rangle \simeq \langle ab, ac, bc \rangle$$

is isotropic. Q.E.D.

Lemma 4.2. *Let f and g be anisotropic quadratic forms. If $f \perp g$ is hyperbolic then $\dim f = \dim g$.*

Proof. Let $f \simeq \langle a_1, \ldots, a_n \rangle$. Suppose $f \perp g \simeq mH$. If $\dim f = n < m$, then

$$mH \simeq \langle a_1, -a_1\rangle \perp \cdots \perp \langle a_n, -a_n\rangle \perp (m-n)H \simeq \langle a_1, \ldots, a_n\rangle \perp g,$$

so, by Witt's cancellation theorem and theorem 2.3, g is isotropic, which is a contradiction. Hence $\dim f \geqq m$. Similarly, $\dim g \geqq m$. Since $\dim f + \dim g = 2m$, we have $\dim f = \dim g$. Q.E.D.

Corollary 4.3. *Let $f \perp g \simeq mH$. If $\dim f > m$, then f is isotropic.*

Proof. Let $g \simeq g_{an} \perp sH$ be the Witt's decomposition of g. If f is anisotropic, then $\dim f = \dim g_{an} \leqq m$ by lemma 4.2. Hence if $\dim f > m$, then f is isotropic. Q.E.D.

Theorem 4.4. *If a 5-dimensional regular quadratic form q is universal then it is isotropic.*

Proof. Suppose that q is a regular quadratic form which is universal. Then q is a group form with $G(q) = \dot{F}$, and $q \simeq \langle 1, a, b, c, d\rangle$ by theorem 2.4. It follows that $q \simeq \langle abcd\rangle \otimes q$ and hence

$$abcd\dot{F}^2 = d(q) = d(\langle abcd\rangle \otimes q) = \dot{F}^2.$$

Therefore $\langle ab\rangle = \langle cd\rangle$. Thus we have the result:

$$q \simeq \langle 1, a, b, c, d\rangle \simeq \langle a\rangle \otimes q \simeq \langle a, 1, ab, ac, ad\rangle \simeq \langle 1, ab, ac, ad, a\rangle$$
$$\simeq \langle 1, cd, ac, ad, a\rangle \simeq \langle 1, ac, ad, cd, a\rangle \simeq \langle\langle ac, ad\rangle\rangle \perp \langle a\rangle.$$

Hence we may assume that $q \simeq \langle\langle a, b\rangle\rangle \perp \langle c\rangle$, so $q \perp \langle ac, bc, abc\rangle \simeq \langle\langle a, b, c\rangle\rangle$. Since q is universal, it follows that the 3-fold Pfister form $\langle\langle a, b, c\rangle\rangle$ is isotropic. From this, $q \perp \langle ac, bc, abc\rangle \simeq 4H$ by theorem 3.1. By corollary 4.3, q is isotropic. Q.E.D.

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