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On the Bounded Multilinear Maps and Multilinear Product Spaces

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1. Introduction

The purpose of this paper is to investigate the properties of bounded multilinear maps and we define a multilinear product and such that multilinear product spaces are normed linear spaces. A vector topology of a topological vector space is determined by a neighborhood system of the origin θ without inducing the concept of metric or norm. We define the bounded set and bounded map by using the concept of neighborhood.

2. Basic concepts

Definition 1.1. A topological vector space is a linear space E with a topology such that addition and scalar multiplication are each continous simultaneously in both variables; more precisely such that each of the following maps is continuous.

(a) the map of the product, $E\times E$ with the product topology, into E, which is given by $(x, y)\to x+y$ for x, y in E;

(b) the map of the product, $K\times E$, of the scalar field K and E, which is given by $(\lambda, x)\to\lambda x$ for λ in K and x in E.

Proposition 1.2. *In a topological vector space E there exists a fundamental system $\mathfrak{N}$ of neighborhood of θ such that:*

(1) *for U in $\mathfrak{N}$ there is a number $V\in\mathfrak{N}$ such that $V+V\subset U$.*

(2) *for $\lambda\in C$ $(\lambda\neq 0)$ and for U, $\lambda U\in\mathfrak{N}$.*

(3) *for $x\in E$ and $U\in\mathfrak{N}$ there is a $Cx>0$ such that $\lambda x\in U$ for $\lambda\in C$ with $|\lambda|\leq Cx$.*

(4) *for $U\in\mathfrak{N}$ there is a member $V\in\mathfrak{N}$ with $V\subset U$ such that $\lambda V\subset V$ for $\lambda\in C$ with $|\lambda|\leq 1$.*

(5) *If E is a Hausdorff space, then $\cap\{U: U\in\mathfrak{N}\}=\{\theta\}$.*

Definition 1.3. A set A in a vector space E over C is balanced if $\lambda A\subset A$ for every $\lambda\in C$ with $|\lambda|\leq 1$.

Proposition 1.4. *If $(E, \mathcal{T})$ is a metrizable topological vector space, then there is a fundamental system of neighborhoods of θ satisfying the following conditions:*

(1) *each U_n is balanced*

(2) $U_1\supset U_2\supset U_3\supset\cdots\cdots\cdots$

(3) $\bigcap_{n=1}^{\infty} U_n=\{\theta\}$

Definition 1.5. A subset B of a metric space (X, d) is said to be bounded if there exist $a\in X$ and $\lambda>0$ such that $d(a, x)<\lambda$ for all $x\in B$.

Definition 1.6. A subset B of a topological vector space E is said to be bounded if for any neighhorhood U of θ there exists $\lambda>0$ such that $B\subset\lambda U$.

3. Bounded multilinear maps

Let $E_1, E_2, \cdots, E_n$ and G be vector spaces over $\boldsymbol{R}$.

Definition 2.1. The operator $m:E_1\times E_2\times\cdots\times E_n\to G$ is said to be multilinear if for any $x_i, x_i'\in E_i$ $(i=1, 2, \cdots, n)$

(1) $m(x_1, x_2, \cdots, x_i+x_i', \cdots, x_n)=m(x_1, x_2, \cdots, x_i, \cdots, x_n)+m(x_1, x_2, \cdots, x_i', \cdots, x_n)$

(2) $m(x_1, x_2, \cdots, \lambda x_i, \cdots, x_n)=\lambda m(x_1, x_2, \cdots, x_i, \cdots, x_n)$

Lemma 2.2. *Let $E_1, E_2, \cdots, E_n$ and G be topological spaces. A multilinear map$(x_1, x_2, \cdots, x_n)\to m(x_1, x_2, \cdots, x_n)$ for $E_1\times E_2\times\cdots\times E_n$ into G is continuous on the product space $E_1\times E_2\times\cdots\times E_n$ if and only if it is continuous at $(\theta, \theta, \cdots, \theta)$.*

Definition 2.3. Let $E_1, E_2, \cdots, E_n$ and G be topological vector spaces. A multilinear map $m:E_1\times E_2\times\cdots\times E_n\to G$ is said to be bounded if for any bounded set $A_1\times A_2\times\cdots\times A_n\subset E_1\times E_2\times\cdots\times E_n$, $m(A_1\times A_2\times\cdots\times A_n)$ is bounded.

Theorem 2.4. *Let $E_1, E_2, \cdots, E_n$ and G be topological vector spaces. A continuous mulitilinear map m: $E_1\times E_2\times\cdots\times E_n\to G$ is bounded.*

Proof. Since $m:E_1\times E_2\times\cdots\times E_n\to G$ is a continuous multilinear map, for any neighborhood W of θ in G, there is a neighborhood U_i of θ in E_i for each $i\in\{1, 2, \cdots n\}$ such that $m(U_1\times U_2\times\cdots\times U_n)\subset W$. Let $A_1\times A_2\times\cdots\times A_n$ be bounded. Then for the neighborhoods $U_1, U_2, \cdots, U_n$, there are $\lambda_1, \lambda_2, \cdots, \lambda_n$ $(\lambda_i>0,\ i=1, 2, \cdots, n)$ such that $A_1\subset\lambda_1 U_1,\ A_2\subset\lambda_2 U_2, \cdots\cdots,\ A_n\subset\lambda_n U_n$.

Hence $m(A_1\times A_2\times\cdots\times A_n)\subset m(\lambda_1 U_1\times\lambda_2 U_2\times\cdots\times\lambda_n U_n)=\lambda_1\lambda_2\cdots\lambda_n\ m(U_1\times U_2\times\cdots\times U_n)\subset\lambda_1\lambda_2\cdots\lambda_n\ W$. Thus m is bounded at θ and hence m is bounded.

Theorem 2.5. *Let $E_1, E_2, \cdots, E_n$ be metrizable topological vector spaces and G be a topological vector space. Then bounded multilinear map $m:E_1\times E_2\times\cdots\times E_n\to G$ is continuous.*

Proof. Let $m:E_1\times E_2\times\cdots\times E_n\to G$ be a bounded multilinear map. Suppose m is not continuous. Then there is a balanced neighborhood W of θ in G such that $m^{-1}(W)$ is not a neighborhood of $(\theta, \theta, \cdots, \theta)$ in $E_1\times E_2\times\cdots\times E_n$. Since $E_1, E_2, \cdots, E_n$ are metrizable topological vector spaces, by Prop. 1.2, there are fundamental systems $\{U_m{}^i;\ m=1, 2, \cdots\}$ of neighborhoods of θ in $E_i(i=1, 2, \cdots, n)$ such that

(a) each $U_m{}^i$ is balanced $(m=1, 2, \cdots)$ in $E_i(i=1, 2, \cdots, n)$

(b) $U_1{}^i\supset U_2{}^i\supset\cdots(i=1, 2, \cdots, n)$

(c) $\bigcap\limits_{m=1}^{\infty} U_m{}^i=\{0\}\quad(i=1, 2, \cdots, n)$

Since $\frac{1}{l_1}U_{l_1}{}^1\times\frac{1}{l_2}U_{l_2}{}^2\times\cdots\times\frac{1}{l_n}U_{l_n}{}^n\not\subset m^{-1}(W)$. We take $(x_{l_1}{}^1,\ x_{l_2}{}^2, \cdots, x_{l_n}{}^n)\in\frac{1}{l_1}U_{l_1}{}^1\times\frac{1}{l_2}U_{l_2}{}^2\times\cdots\times\frac{1}{l_n}U_{l_n}{}^n\sim m^{-1}(W)$. Then the sequence $\{l_1x_{l_1}{}^1,\ l_2x_{l_2}{}^2, \cdots, l_nx_{l_n}{}^n\}$ converges to $(\theta, \theta, \cdots, \theta)$ and hence $\{l_1x_{l_1}{}^1, l_2x_{l_2}{}^2, \cdots, l_nx_{l_n}{}^n\}$ is bounded.

Since m is bounded, there is a $\lambda>0$ such that $m(l_1x_{l_1}{}^1, l_2x_{l_2}{}^2, \cdots, l_nx_{l_n}{}^n)=l_1l_2\cdots l_n\ m(x_{l_1}{}^1, x_{l_2}{}^2, \cdots, x_{l_n}{}^n)\in\lambda W$.

Let $l_1, l_2, \cdots, l_n \geq \lambda$. Then $m(x_{l_1}{}^1, x_{l_2}{}^2, \cdots, x_{l_n}{}^n) \in \frac{\lambda}{l_1 l_2 \cdots l_n} W \subset W$. But $(x_{l_1}{}^1, x_{l_2}{}^2, \cdots, x_{l_n}{}^n) \notin m^{-1}(W)$ which is a contradiction. Thus m is a continuous multilinear map.

Corollary 2.6. *Let $E_1, E_2, \cdots, E_n$ and G be normed linear spaces. A multilinear operator $m: E_1 \times E_2 \times \cdots \times E_n \to G$ is continuous if and only if it is bounded.* From now on, n is even.

Definition 2.7. Let X be a real vector space. A multilinear product ($m.p.$) on X is a real function $\langle x_1, x_2, \cdots, x_n \rangle$ on $X \times X \times \cdots \times X$ with the following properties.

(1) $\langle x_1, x_2, \cdots, x_i + x_i', \cdots, x_n \rangle = \langle x_1, x_2, \cdots, x_i, \cdots, x_n \rangle + \langle x_1, x_2, \cdots, x_i', \cdots x_n \rangle$ $(i=1, 2, \cdots, n)$

(2) $\langle x_1, x_2, \cdots, \lambda x_i, \cdots, x_n \rangle = \lambda \langle x_1, x_2, \cdots, x_n \rangle$ $(i=1, 2, \cdots, n)$

(3) $\langle x, x \cdots, x \rangle > 0$ for $x \neq \theta$

(4) $|\langle x_1, x_2, \cdots, x_n \rangle|^n \leq \langle x_1, \cdots, x_1 \rangle \langle x_2, \cdots, x_2 \rangle \cdots \langle x_n, \cdots, x_n \rangle$ for all $x_1, x_i, \cdots, x_n, x_n'$ in X and for all real number λ. A vector space with a m.p. is called a multilinear product space (m.p. space).

Theorem 2.8. *m.p. space is a normed linear space X with* $|||x||| = \langle x, x, \cdots, x \rangle^{\frac{1}{n}}$.

Proof. (1) Let us show that $|||\cdot|||$ is subadditive, i.e., for all $x, y \in X$, $|||x+y||| \leq |||x||| + |||y|||$.

$|||x+y|||^n = \langle x+y, x+y, \cdots, x+y \rangle = \langle x, x, \cdots, x \rangle + \langle y, x, \cdots, x \rangle + \cdots + \langle x, \cdots, y \rangle + \langle y, y, x, \cdots, x \rangle + \cdots + \langle y, y, \cdots, y \rangle \leq |||x|||^n + \frac{n!}{1!(n-1)!} \langle x, x, \cdots, x \rangle^{n-1} \langle y, \cdots, y \rangle + \frac{n!}{2!(n-2)!} \langle x, x, \cdots, x \rangle^{n-2} \langle y, \cdots, y \rangle + \cdots + \frac{n!}{(n-1)!1!} \langle x, \cdots, x \rangle \langle y, \cdots, y \rangle^{n-1} + |||y|||^n = |||x|||^n + \frac{n!}{1!(n-1)!} (|||x|||^{n-1} |||y|||) + \frac{n!}{2!(n-2)!} (|||x|||^{n-2} \times |||y|||^2) + \cdots + \frac{n!}{(n-1)!1!} (|||x||| \, |||y|||^{n-1}) + |||y|||^n = (|||x||| + |||y|||)^n$. Thus $|||x+y||| \leq |||x||| + |||y|||$.

(2) Let us show that $|||\cdot|||$ is positively homogeneous of degree 1, i.e., for all $x \in X$, and all $\lambda \in \boldsymbol{R}$ $|||\lambda x||| = |\lambda| \, |||x|||$;

$|||\lambda x|||^n = \langle \lambda x, \lambda x, \cdots \lambda x \rangle = \lambda^n \langle x, x, \cdots, x \rangle = \lambda^n |||x|||^n$. Hence $|||\lambda x||| = |\lambda| \, |||x|||$.

(3) Let's show that $x \in X$, $|||x||| = 0$ implies $x \in \theta$. Assume $x \neq \theta$. Then $\langle x, x \cdots, x \rangle > 0$ and hence $|||x||| > 0$, i.e., $|||x||| \neq 0$.

If X and Y are m.p. spaces, then $X \oplus Y = \{(x, y) : x \in X, y \in Y\}$ is a m.p. space with componentwise addition, scalar multiplication together with the m.p. defined by $\langle (x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n) \rangle = \langle x_1, x_2, \cdots, x_n \rangle + \langle y_1, y_2, \cdots, y_n \rangle$. The norm on $X \oplus Y$ is given by $|||(x, y)||| = (|||x|||^n + |||y|||^n)^{\frac{1}{n}}$. If T_1 and T_2 are bounded linear operators on m.p. space X and Y respectively, then the bounded linear operator $T_1 \oplus T_2$ on $X \oplus Y$ is defined by $(T_1 \oplus T_2)(x, y) = (T_1 x, T_2 y)$.

Notation. $W(T) = \{\langle \underbrace{Tx, Tx, \cdots, Tx}_{\frac{n}{2}}, \underbrace{x, x, \cdots, x}_{\frac{n}{2}} \rangle : |||x||| = 1\}$.

Theorem 2.9. *Let A and B be bounded linear operators on m.p. spaces X and Y respectively. If $W(A)$ and $W(B)$ are convex subsets of C, then $W(A \oplus B) = Co(W(A) \cup W(B))$* where $Co(X)$ *denotes the convex hull of the set X.*

Proof. Let $\lambda \in W(A \oplus B)$. We can find an element (x, y) in $X \oplus Y$ such that $|||(x, y)||| = (|||x|||^n + |||y|||^n)^{\frac{1}{n}} = 1$ and $\lambda = \langle \underbrace{(A \oplus B)(x, y), \cdots, (A \oplus B)(x, y)}_{\frac{n}{2}}, \underbrace{(x, y), \cdots, (x, y)}_{\frac{n}{2}} \rangle = \langle Ax, \cdots, Ax, x, \cdots, x \rangle + \langle By, \cdots, By, y, \cdots, y \rangle$. Putting $|||x|||^n = \alpha$, we see that $0 \leq \alpha \leq 1$ and $|||y|||^n = 1 - \alpha$. Notice that $\lambda \in W(B)$ for $\alpha = 0$ and $\lambda \in W(A)$ for $\alpha = 1$. If $0 < \alpha < 1$, then $\lambda = \alpha \langle Ax, \cdots, Ax', Ax', x', \cdots, x' \rangle + (1-\alpha) \langle By', \cdots,$

$By', y', \cdots, y'\rangle$ where $x' = \frac{x}{\sqrt[n]{\alpha}}$ and $y' = \frac{y}{\sqrt[n]{1-\alpha}}$ are unit vectors in X and Y respectively. This shows that $\lambda \in Co(W(A) \cup W(B))$. Conversely suppose $\lambda \in Co(W(A) \cup W(B))$ so that $\lambda = \beta\mu + (1-\beta)\nu$ with $0 \leq \beta \leq 1$, $\mu \in W(A)$ and $\nu \in W(B)$. There exist unit vectors x in X and y in Y such that $\mu = \langle Ax, \cdots, Ax, x, \cdots, x\rangle$ and $\nu = \langle By, \cdots, By, y, \cdots, y\rangle$. Then $\lambda = \beta\langle Ax, \cdots, Ax, x, \cdots, x\rangle + (1-\beta)\langle By, \cdots, By, By, y, \cdots, y\rangle = \langle A\sqrt[n]{\beta}x, \cdots, A\sqrt[n]{\beta}x, \sqrt[n]{\beta}x, \cdots, \sqrt[n]{\beta}x\rangle + \langle B\sqrt[n]{1-\beta}y, \cdots, B\sqrt[n]{1-\beta}y, \sqrt[n]{1-\beta}y, \cdots, \sqrt[n]{1-\beta}y\rangle = \langle(A\sqrt[n]{\beta}x, B\sqrt[n]{1-\beta}y), \cdots, (A\sqrt[n]{\beta}x, B\sqrt[n]{1-\beta}y), (\sqrt[n]{\beta}x, \sqrt[n]{1-\beta}y), \cdots, \sqrt[n]{\beta}x, \sqrt[n]{1-\beta}y)\rangle = \langle(A \oplus B)(\sqrt[n]{\beta}x, \sqrt[n]{1-\beta}y), \cdots, (A \oplus B)(\sqrt[n]{\beta}x, \sqrt[n]{1-\beta}y), (\sqrt[n]{\beta}x, \sqrt[n]{1-\beta}y), \cdots, (\sqrt[n]{\beta}x, \sqrt[n]{1-\beta}y)\rangle$. Since $|||(\sqrt[n]{\beta}x, \sqrt[n]{1-\beta}y)||| = 1$, we conclude that $\lambda \in W(A \oplus B)$.

Theorem 2.10. *Let X and Y be m.p. spaces. Suppose $S: X \to X$ and $T: T \to Y$ are bounded linear operators. Then*

(a) *$S \oplus T$ is bounded from below if and only if S and T are both bounded from below:*

(b) *$\overline{R(S \oplus T)} = X \oplus Y$ if and only if $\overline{R(S)} = X$ and $\overline{R(T)} = Y$, where $R(U)$ denote the range of the operator U.*

Proof. (a) Suppose $S \oplus T$ is bounded from below. Then there exists $m > 0$ such that $|||(S \oplus T)(x, y)||| \geq m |||(x, y)|||$ for all (x, y) in $X \oplus Y$. This gives (*) $|||Sx|||^n + |||Ty|||^n \geq m^n(|||x|||^n + |||y|||^n)$ for all x in X and y in Y. Taking $y = 0$ in (*), we see that $|||Sx||| \geq m|||x|||$ for all x in X showing that S is bounded from below. Similary T is bounded from below. Conversely suppose that S and T are bounded from below. Then there exist positive constants m_1 and m_2 such that $|||Sx||| \geq m_1 |||x|||$ for all x in X and $|||Ty||| \geq m_2 |||y|||$ for all y in Y. If $m = \min(m_1, m_2)$ then it is easy to verify that $|||(S \oplus T)(x, y)||| > m |||(x, y)|||$ for all (x, y) in $X \oplus Y$.

(b) Follows from the fact that $(x_n, y_n) \to (x, y)$ in $X \oplus Y$ if and only if $x_n \to x$ in X and $y_n \to y$ in Y.

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