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A Consideration on the Relation of Proclusions and α-Spaces.

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1. Preliminaries

In the study of topological spaces, the quotient map is considered as a typical universal map and useful tool because of the universal factorization property.

In 1967, the concept of "Proclusion" has been proposed by N.E. Steenrod, [3]. Thereafter, O. Wyler defined proclusions by generalizing quotient maps in 1973, as definition 2.1 of this article.

The purpose of this study is to systemize the properties of proclusions. Furthermore, we introduce α-spaces and investigate the relation between α-spaces and proclusions.

To simplify the discussion, we use the following notations.

Notations

i) *TOP*: the category of topological spaces and continuous maps.

ii) *SET* : the category of sets and functions.

iii) $\mathcal{T}$: a nontrivial epireflective subcategory of *TOP*.

iv) A functor $P: TOP \to SET$ is a forgetful functor.

v) *corh* $\mathcal{A}$: the full and replete subcategory of $\mathcal{T}$ consists of α-spaces.

2. Properties of Proclusions

Here we introduce some definitions and theorems which will be useful in the sequal.

Definition 2.1. Let $P: TOP \to SET$ be a *forgetful functor*.

If X in an object of *TOP* such that $f: X \to Y$ and u is an morphism of *SET* such that domain of u is PX then (f, v) is a P-proclusion pair at X over u, where v is a morphism of *SET* with the following two properties:

(i) $PF = vu$ in *SET*

(ii) Whenever domain of g is X and t is a morphism of *SET* such that $Pg = tu$ in *SET*, then h is uniquely exists in *TOP* such that $g = hf$ in *TOP* and $t = (Ph)v$ in *SET*.

Definition 2.2. If $(f, \mathrm{id}PY)$ is a P-proclusion pair over Pf, then $f: X \to Y$ in *TOP* is a P-proclusion.

Here, we can obtain another definition by categorical duality, that is P-inclusion pair and P-inclusion. Precisely speaking, $f: X \to Y$ of *TOP* is a P-inclusion if for $g: Z \to Y$ in *TOP* and t in *SET* such that $Pg - (Pf)t$ in *SET*, $h: Z \to X$ is uniquely exists in *TOP* such that $g = fh$ in *TOP* and $Ph = t$ in *SET*.

Theorem 2.3. *If $f: X \to Y$ in TOP is a P-proclusion and Pf is epimorphism in SET, then f is epimorphism in SET, i.e., functor P always reflects epimorphism,*

Proof. If $f:X\to Y$ is a P-proclusion and Pf epimorphic in SET, then $^{\exists}Pf:PX\to PY$ such that $\forall Px\in PX$, there is $Py\in PY$ with $Pf(Px)=Py$. So, $Pf(Px)=P[f(x)]=Py$, i.e., $\forall x\in X$, $\exists y\in Y$ with $f(x)=y$. Hence f is epimorphism in TOP.

In consequence, we obtain the following corollary.

Corollary 2.4. *If $f\in TOP$ is P-inclusion and $Pf\in SET$ is monomorphism, then f is monomorphism, i.e., P always reflects monomorphism.*

Theorem 2.5. (a) *Every isomorphism of TOP is a P-proclusion.*

(b) *Conversely, if (f, v) is a P-proclusion pair over an isomorphism u of SET, then f is an isomorphism of TOP and v an isomorphism of SET.*

Proof. (a) Assume that (f, v) is a P-proclusion pair over u and f is an isomorphism of TOP. Then $\exists 1 f':Y\to X$ such that $f\cdot f'=\mathrm{id}Y$, $f'\cdot f=\mathrm{id}Y$.

We shall show that $(f, \mathrm{id}PY)$ is a P-proclusion pair over Pf, i.e., (i) $Pf=(\mathrm{id}PY)(Pf)$ (ii) $\forall g\in TOP$, $t\in SET$, such that $Pg=t(Pf)$ in $SET\Rightarrow \exists 1 h\in TOP$ such that $g=hf$ in TOP and $t=(Ph)(\mathrm{id}PY)$ in SET.

Since $\mathrm{id}PY=P(\mathrm{id}Y)=P(f\cdot f')=(Pf)(Pf')$, then $(\mathrm{id}PY)(Pf)=[(Pf)(Pf')](Pf)=Pf$.

By the property of P-proclusion pair (f, v) over u, we let $Pg=tu$. Since $(\mathrm{id}PY)(Pf)=Pf=vu$, $\mathrm{id}PY=v$ and $Pf=u$.

Hence $Pg=tu=t(Pf)$ and $t=(Ph)v=(Ph)(\mathrm{id}PY)$.

(b) Assume that (f, v) is a P-proclusion pair over u and $u:PX\to S$ is an isomorphism is SET. Then $\exists u':S\to PX$ such that $u\cdot u'=\mathrm{id}S$, $u'\cdot u=\mathrm{id}PX$.

Let $u=V\cdot(Pf)$ then $u'u=u'(V(Pf))=(u'V)(vu)=u'(Vv)u=(u'u)(Vv)$, i.e., $Vv=\mathrm{id}S$.

Hence $\exists 1 v'(=V):PY\to S$.

Since $Pf=vu$ and v and u are isomorphism of SET, Pf is isomorphism and hence f is isomorphism of TOP.

We may have following question.

"Does P-proclusion always exist in many topological situation?" Similarly, "Does P-inclusion always exist in many topological situation?" The answer is easily obtained by the following. That is, in order to $f:X\to Y$ and $g=hf:X\to Z$ are continuous, Y have to the finest topology for the given topology of X. Similarly, f is a P-inclusion if and only if X has the coarsest topology for the given topology of Y.

On the otherhand, by Theorem 2.3 and corollary 2.4, we may obtain general relation that the proclusions correspond to quotient spaces and quotient algebras, the inclusions correspond to subspaces and subalgebras.

Definition 2.6. A morphism e is called an *extremal epimorphism* if

(i) e is epimorphism

(ii) extremal condition: Whenever $e=mf$ and m is monomorphism, then m is isomorphism.

Definition 2.7. A morphism e is called a *strong epimorphism* if

(i) e is epimorphism,

(ii) whenever $mu=ve$ and m is monomorphism, there is a unique morphism t such that $v=mt$ and $u=te$.

We note that a strong epimorphism is an extremal epimorphism but the converse is not always true.

Of course, extremal monomorphism and strong monomorphism are defined dually.

The following Theorem 2.8 whose proof is in [4] is useful to prove the main result of this paper.

Theorem 2.8. *Assume that P has the following properties:*

(a) *P preserves and reflects monomorphisms.*

(b) *Every morphism g of TOP has a factorization $g=g_1g_2$ such that g_1 is monomorphism in TOP and Pg_2 is a strong epimorphism in SET.*

(c) *For $f:X\to Y$ of TOP and strong epimorphism u of SET such that domain of u is PX, there is a P-proclusion pair (f,v) at X over u.*

Then the following statements are equivalent.

(i) *f is an extremal epimorphism of TOP.*

(ii) *f is a strong epimorphism of TOP.*

(iii) *f is a P-proclusion, and Pf is a strong epimorphism of SET.*

Definition 2.9. A subcategory $\mathcal{T}$ of TOP is called a full subcategory of TOP if for all X, Y of TOP, $\hom_{\mathcal{T}}(X,Y)=\hom_{TOP}(X,Y)$.

Definition 2.10. A subcategory $\mathcal{T}$ of TOP is called a replete subcategory (or, isomorphism closed subcategory) if $f:X\to Y$ is an isomorphism of TOP and X is an object of $\mathcal{T}$, then f is a morphism in $\mathcal{T}$ and Y is an object of $\mathcal{T}$.

Definition 2.11. A morphism $\eta_X:X\to RX$ of TOP is called a $\mathcal{T}$-universal in TOP if RX is an object of $\mathcal{T}$, and for isomorphism $f:X\to Y$ of TOP such that Y is an object of $\mathcal{T}$, there is a unique morphism $\bar{f}:RX\to Y$ in $\mathcal{T}$ such that $f=\bar{f}\eta_X$ in TOP.

Definition 2.12. A subcategory $\mathcal{T}$ is reflective in TOP if for every object X of TOP, $\mathcal{T}$-universal morphism $\eta_X:X\to RX$ exists in TOP.

It is well known that $\mathcal{T}$-couniversal morphisms and coreflective subcategories are dual to $\mathcal{T}$-universal morphism and reflective subcategories.

3. α-Spaces

In this chapter, $\mathcal{T}$ is considered as a non-trivial epireflective subcategory of TOP, i.e., $\mathcal{T}$-universal map $\eta_X:X\to RX$ of TOP is epimorphism and every object of $\mathcal{T}$ is not indiscrete space.

Definition 3.1. Let $\mathcal{A}$ be a class of spaces which contains at least one non-empty space.

For any space $X\in TOP$, $\mathcal{A}/X=\{u\mid u:A\to X$ continuous map, $\forall A\in\mathcal{A}\}$.

Definition 3.2. The object αX of $\mathcal{T}$ is determined by X, which is consist of underlying set $|X|$ of X such that every $u\in\mathcal{A}/X$ is continuous and X has the finest topology in $\mathcal{T}$.

Definition 3.3. X is called an α-space if $\alpha X=X$.

Theorem 3.4. *For every space X, the map $id|X|:\alpha X\to X$ is couniversal for the category corh $\mathcal{A}$ of α-spaces.*

Proof. See [5].

In the condition of Theorem 3.4, since $id|X|:\alpha X\to X$ is couniversal, for $f:X\to Y$, we have $f:\alpha X\to\alpha Y$ in $\mathcal{T}$.

Now, we will denote $f=\alpha f:\alpha X\to\alpha Y$, then α becomes a functor from $\mathcal{T}$ to *corh* $\mathscr{A}$.

In other words, α is a right adjoint to the embedding functor from *corh* $\mathscr{A}$ to $\mathcal{T}$. Hence the functor α is a coreflector for *corh* $\mathscr{A}$ and *corh* $\mathscr{A}$ is coreflective subcategory of $\mathcal{T}$.

4. Main Results

By using previous results we obtain the following concerning the relation between proclusions and α-spaces.

For convenience, *corh* $\mathscr{A}=\mathscr{A}^*$

Theorem 4.1. *Assume that (f, v) is a P-proclusion pair over u such that X is an object of $\mathscr{A}^*$ and $f:X\to Y$ is in $\mathcal{T}$ and $\varepsilon_Y:RY\to Y$ is $\mathscr{A}^*$-couniversal such that $f=\varepsilon_Y\cdot\bar{f}$ in $\mathcal{T}$ for $\bar{f}:X\to RY$ in $\mathscr{A}^*$.*

If for any w of SET, $(P\varepsilon_Y)w=v$ and $\alpha\bar{f}=wu$ in SET, then ε_Y is an isomorphism of $\mathcal{T}$, (f, v) is an α-proclusion pair over u and Y is α-space.

Proof. By assumption, since (f, v) is a P-proclusion pair over u $\exists 1 h: Y\to RY$ in $\mathcal{T}$ such that $\bar{f}=hf$ in $\mathcal{T}$ and $(Ph)v=u$ in SET. So, $(\varepsilon_Y h)f=f:X\to Y$ and $P(\varepsilon_Y h)v=v:S\to PY$, whenever $u:PX\to S$. Hence $\varepsilon_Y h=idY$. But $h\varepsilon_Y:RY\to RY$ in $\mathscr{A}^*$. Since $\varepsilon_Y(h\varepsilon_Y)=(\varepsilon_Y h)\varepsilon_Y=\varepsilon_Y$, $h\varepsilon_Y=idRY$, thus ε_Y is an isomorphism.

Since $\mathscr{A}^*$ is replete, $\varepsilon_Y:RY\to Y$ is in $\mathscr{A}^*$ and $f:X\to Y$ is in $\mathscr{A}^*$.

By hypothesis, X is α-space. Hence (f, v) is an α-proclusion pair over u and Y is α-space too.

In particular, a surjective proclusion is called a quotient map.

Theorem 4.2. *The strong epimorphism in $\mathcal{T}$ and corh $\mathscr{A}$ are quotient map.*

Proof. Let f be a strong epimorphism in $\mathcal{T}$ (or, *corh* $\mathscr{A}$).

By Theorem 2.8, f is P-proclusion and Pf is strong epimorphism iff f is strong epimorphism.

Hence f is surjective proclusion, i.e., quotient map.

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