韓國數學教育學會誌<數學教育>
1984. 12, Vol. XXIII, No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1984, Vol. XXIII, No. 1

On f-Proximinal and f-Remotal Points of Pairs of Sets

by T.D. Narang*
Guru Nanak Dev University, India

In this paper we introduce the notions of f-proximinal points and f-remotal points of pairs of sets in a Hausdorff topological space X relative to a functional f on $X\times X$ and discuss the existence of such points.

Let X be a Hausdorff topological space, f a continuous real-valued function on $X\times X$ and U, V a pair of subsets of X. We call points $u^*\in U$, $v^*\in V$ ***f-proximinal points*** of sets U, V if $f(u^*, v^*)=\inf\{f(u,v): u\in U, v\in V\}$ and ***f-remotal points*** of sets U, V if $f(u^*, v^*)=\sup\{f(u,v): u\in U, v\in V\}$. When one of the two sets is reduced to a single point, the problem of f-proximinal points of pair of sets is reduced to that of f-best approximation (cf. [8] or [4]) and of f-remotal points of pair of sets to that of f-farthest points (cf. [2] or [5]).

The set U is said to be ***inf-compact*** with respect to (w.r.t.) the set V if each minimizing net of V (i.e. a net $\{u_\alpha\}$ such that $f(u_\alpha, V)\longrightarrow\inf\{f(u,v): u\in U, v\in V\}$, where $f(x, V)=\inf\{f(x,v): v\in V\}$) contains a subnet converging in U.

The set U is said to be ***sup-compact*** w.r.t. the set V if each maximizing net of V (i.e. a net $\{u_\alpha\}$ such that $f_V(u_\alpha)\longrightarrow\sup\{f(u,v): u\in U, v\in V\}$, where $f_V(x)=\sup\{f(x,y): y\in V\}$) contains a subnet converging in U.

The set V is said to be ***f-proximinal*** w.r.t. U if each point x of U has f-best approximation $\bar{v}$ in V i.e. $\bar{v}\in V$ satisfying $f(x,\bar{v})=\inf\{f(x,y): y\in V\}$. V is said to be ***f-remotal*** w.r.t. U if each point x of U has f-farthest point $\bar{v}$ in V i.e. $\bar{v}\in V$ satisfying $f(x,\bar{v})=\sup\{f(x,y): y\in V\}$.

The following theorem gives the existence of f-proximinal points of pair of sets U, V.

Theorem 1. *Let U, V be a pair of closed subsets of the space X such that U is inf-compact w.r.t. V and V is f-proximinal w.r.t. U. Then f-proximinal points of the pair of sets U, V exist.*

Proof. Let $f(U,V)=\inf\{f(x,y): x\in U, y\in V\}=\inf\{f(x,V): x\in U\}$. Then there exist a net $\{u_\alpha\}$ in U such that $f(u_\alpha, V)\longrightarrow f(U,V)$. Since U is inf-compact w.r.t. V, this net $\{u_\alpha\}$ has a convergent subnet $\{u_\beta\}\longrightarrow u^*\in U$. Since the function $x\longrightarrow f(x,V)$ is continuous, it follows that $f(u^*, V)=f(U,V)$. Since V is f-proximinal w.r.t. U, there exists an element $v^*\in V$ such that $f(u^*, v^*)=f(u^*, V)=f(U,V)$. This implies that u^*, v^* are f-proximinal points of the pair U, V.

Since inf-compact sets are f-proximinal [4], we have:

Corollary 1. *Let U, V be a pair of closed subsets of X such that U is inf-compact w.r.t. V and V is inf-compact w.r.t. U then f-proximinal points of the pair U, V exist.*

* The author is thankful to the U.G.C. for financial support.

Corollary 2. *Let U, V be a pair of closed subsets of X such that U is compact and V is f-proximinal w.r.t. U then f-proximinal points of the pair U, V exist.*

The following theorem gives the existence of f-remotal points of the pair of sets in a Hausdorff topological vector space X.

Theorm 2. *Let U, V be a pair of bounded subsets of the space X such that U is sup-compact w.r.t. V and V is f-remotal w.r.t. U. Then f-remotal points of the pair of sets U, V exist.*

The proof of this theorem is analogous to that of Theorem 1.

Since sup-compact sets are f-remotal [5], we have:

Corollary 3. *Let U, V be a pair of bounded subsets of X such that U is sup-compact w.r.t. V and V is sup-compact w.r.t. U. Then f-remotal points of the pair U, V exist.*

Corollary 4. *Let U, V be a pair of bounded subsets of X such that U is compact and V is f-remotal w.r.t. U. Then f-remotal points of the pair U, V exist.*

Remark 1. If the sets U, V are such that the functional f attains its infimum (supremum) on $U \times V$ then f-proximinal points (f-remotal points) of the pair U, V exist.

Remark 2. In case X is a metric space and $f=d$, the metric on X, the notion of f-proximinal points of the pair of sets coincides with that of proximinal points (cf. [3]), of f-remotal points of the pair of sets with that of farthest points relative to the two sets (cf. [1]) and results proved in this paper generalize some of the earlier known results of [1] and [3].

Remark 3. Some characterizations of f-proximinal points of pairs of convex sets were given by D.V. Pai [6] and by D.V. Pai and P. Govindarajulu [7] in Hausdorff locally convex linear topological spaces.

References

1. G.C. Ahuja, T.D. Narang and Swaran Trehan, On farthest points, *J. Indian Math. Soc.*, 39 (1975), 293-297.
2. P. Govindarajulu and D.V. Pai, On f-farthest points of sets, *Indian J. Pure Appl. Math.*, 14 (1983), 873-882.
3. T.D. Narang, On distance sets, *Indian J. Pure Appl. Math.*, 7(1976), 1137-1141.
4. T.D. Narang, On f-best approximation in topological spaces, communicated.
5. T.D. Narang, On f-farthest points in topological spaces, communicated.
6. D.V. Pai, Multioptimum of a convex functional, *J. Approximation Theory*, 19 (1977), 83-99.
7. D.V. Pai and P. Govindarajulu, On f-proximinal points of pairs of sets, *Optimizing Methods in Statistics*, Academic Press (1979), 367-384.
8. D.V. Pai and P. Veermani, Applications of fixed point theorems to problems of optimization and best approximation, *Non-linear Analysis and Applications*, Marcel Dekker Inc., New York (1982), 393-400 (Ed. S.P. Singh and J.H. Burry).