A Property of Restricted Lie Algebra

by Byung-mun Choi

1. Introduction

A Lie algebra $FL$ is called a free Lie algebra on $X=\{x_1,\ldots, x_n\}$ if, given any mapping $\phi$ of $X$ into a Lie algebra $M$, there exists a unique homomorphism $\psi: FL \to M$ extending $\phi$.

This algebra $FL$ is constructed by the Lie algebra $F(L)$ generated by $X$.

Here $F(L)$ is the free algebra generated by $X$ with the bracket operation $[a, b]=ab-ba$ for each $a, b \in F(L)$.

The Friedrichs’ theorem is a useful criterion for Lie elements in study of the free Lie algebra over a field of characteristic 0.

This paper will introduce an analogue of criterion in case of characteristic $p \neq 0$.

2. Preliminaries

**Lemma 1** (Friedrichs). Let $F=k(x_1,\ldots, x_n)$ be the free algebra generated by the $x_i$ over a field of characteristic 0.

Let $\delta$ be the diagonal mapping of $F$, i.e., the homomorphism of $F$ into $F \otimes F$ such that $x_i \delta = x_i \otimes 1 + 1 \otimes x_i$.

Then $\alpha \in F$ is a Lie elements, i.e., $\alpha \in FL$ if and only if $\alpha \delta = \alpha \otimes 1 + 1 \otimes \alpha$.

**Definition.** A restricted Lie algebra $L$ of characteristic $p \neq 0$ is a Lie algebra with the operation $\alpha \to \alpha^{(p)}$ ($\alpha \in L$) satisfying the following three condition.

R1) $\forall \alpha \in k$, $\forall \alpha \in L$, $(\alpha \alpha)^{(p)} = \alpha \alpha^{(p)}$.

R2) $(a+b)^{(p)} = a^{(p)} + b^{(p)} + \sum_{i=1}^{p-1} S_i(a, b)$

where $iS_i(a, b)$ is the coefficient of $\lambda^{-1}$ in $a(\lambda a + b)(\lambda a + b)^{p-1}$ ($\lambda$ an indeterminant).

R3) $[a, b^{(p)}] = a(\alpha \beta)^{(p)}$.

**Lemma 2.** Let $FL$ be the free Lie algebra generated by a set $X=\{x_1,\ldots, x_n\}$. If we define $a^{(p)} = a^p$ for every $a \in FL$, then $FL$ is a restricted Lie algebra.

**proof.** R1), R2) are easily verified with some rigorous but elementary calculation.

In $FL$, $(a, b^{(p)}) = ab^p - b^p a$ and

$a(\alpha \beta)^{(p)} = ab^p - b^p a + \sum_{i=1}^{p-1} S_i(a, b)$.

The last term of the right in second equation is 0.

Thus R3) is also hold.
3. Main Theorem

Theorem. Let $F=k(x_1,\ldots,x_n)$ be the free algebra over a field $k$ of characteristic $p\neq 0$. Let $\delta$ be as in Lemma 1.

Then $a \in F$ is in the restricted Lie algebra generated by the $x_i$ if and only if $a \delta = a \otimes 1 + 1 \otimes a$.

Proof. $(a \otimes 1 + 1 \otimes a, b \otimes 1 + 1 \otimes b) = [ab] \otimes 1 + 1 \otimes [ab]$ implies that the set elements $a$ satisfying $a \delta = a \otimes 1 + 1 \otimes a$ is a subalgebra of $F(L)$.

This includes the $x_i$, hence it contains $FL$.

Let $y_1, y_2, \ldots$ be a basis for $FL$. Since $F$ is the universal enveloping algebra of $FL$, the elements $y_1^{k_1} y_2^{k_2} \cdots y_n^{k_n}$, $k_i \geq 0$ form a basis for $F$. Hence the products $(y_1^{k_1} y_2^{k_2} \cdots y_n^{k_n}) \otimes (y_1^{l_1} y_2^{l_2} \cdots y_n^{l_n})$

form a basis for $F \otimes F$.

$(y_1^{k_1} y_2^{k_2} \cdots y_n^{k_n}) \delta = (y_1 \otimes 1 + 1 \otimes y_1)^{k_1} (y_2 \otimes 1 + 1 \otimes y_2)^{k_2} \cdots (y_m \otimes 1 + 1 \otimes y_m)^{k_m}

= y_1^{k_1} y_2^{k_2} \cdots y_n^{k_n} \otimes 1 + k_1 y_1^{k_1-1} y_2^{k_2} \cdots y_n^{k_n} \otimes y_1$

$+ k_2 y_1^{k_1} y_2^{k_2-1} \cdots y_n^{k_n} \otimes y_2 + \ldots + k_m y_1^{k_1} \cdots y_n^{k_n-1} \otimes y_m + (*)$

where $(*)$ is a linear combination of base elements of the form

$y_1^{l_1} y_2^{l_2} \cdots y_i^{l_i-1} y_i^{l_i} y_1^{l_1} y_2^{l_2} \cdots y_n^{l_n}$ with $\sum l_i > 1$.

In order that $a$ shall be a linear combination of the base elements of the form $y_1^{k_1} \cdots y_n^{k_n} \otimes 1$ and $1 \otimes y_1^{l_1} \cdots y_i^{l_i}$, it is necessary that in the expression for $a$ in terms of the only base elements $y_1^{k_1} \cdots y_n^{k_n}$ with one $k_i = 1$ and all the other $k_i = 0$ or $np$.

This means that $a$ is a linear combination of the $y_i$ and $y_1^{np}$.

But $y_1^{np} = (y_1^{p})^n \in FL$. Hence $a \delta = a \otimes 1 + 1 \otimes a$ if and only if $a \in FL$ as a restricted Lie algebra.

References