Integral Representation of $C^\infty$ Solutions of Linear Partial Differential Equations with the Canonical Form

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1. Introduction

In this paper $\Omega$ will denote an open neighborhood $|(x,t)| |x| < r$, $|t| < \delta$ of the origin of $\mathbb{R}^2$.

We consider the general linear P.D.E with the canonical form

$$L = \partial_t + ib(x,t)\partial_x$$

where $b(x,t)$ is a real valued $C^\infty$ function in $\Omega$.

The linear P.D.E $L$ is said to satisfy the condition $(P)$ in if for any $x \in (-r,r)$, the function $t \to b(x,t)$ does not change sign and satisfy condition $(P_1)$ if

(i) $b(x,t) > 0$ for any $(x,t) \in \Omega$ with $x \neq 0$

(ii) $b(0,t) = 0$ for any $t$, $|t| < \delta$ and satisfy $(P_2)$

if $b(x,t) \neq 0$ for any $(x,t) \in \Omega$.

We now assume that $L$ satisfies $(P_1)$ or $(P_2)$. It implies that $L$ satisfies $(P)$. That $Lu = f$ is locally solvable follows immediately from the general criteria for the local solvability of a linear P.D.E due to Nirenberg-Treves [2].

We also assume $L$ satisfies the followings:

$$Lz = \partial_t + ib(x,t)\partial_x = 0,$$

$$\text{Re } z_x > 0$$

has a $C^\infty$ solution in $\Omega$.

We shall represent a $C^\infty$ solution in an integral form in a neighborhood of the origin. When $b(x,t)$ is real analytic, the same result is established in Treves[3].

2. Integral Representation

Let $z = z(x,t)$ be a $C^\infty$ solution of

$$Lz = \partial_t + ib(x,t)\partial_x = 0,$$

$$\text{Re } z_x > 0$$

in $\Omega$. This is the generalization of $x - it^2/2$ of the Mizohata operator.

We write $z(x,t) = \xi(x,t) + \eta(y,t)$

where $\xi$ and $\eta$ are real valued. Thus $\partial \xi/\partial x \neq 0$ in $\Omega$. Therefore we have the right to change variables

$$y = \xi(x,t), \quad s = t$$

in $\Omega$.

Let $z = y + i\phi(y,s)$

where $\phi(y,s) = \eta(x,t)$ real valued, $C^\infty$ and $\partial y/\partial x \neq 0$ in $\Omega$. 

35
In \((y, s)\) coordinates we have
\[
L = \partial / \partial t + ib(x, t) \partial / \partial x \\
= \partial / \partial s + \{ \partial y / \partial t + ib(x, t) \partial y / \partial x \} \partial / \partial y \\
= \partial / \partial s + \lambda(y, s) \partial / \partial y.
\]
But \(Lz = 0\), that is,
\[
0 = L(y + i\phi) = i\phi_t + \lambda(1 + i\phi).
\]
From this,
\[
\lambda = \partial y / \partial t + ib\partial y / \partial x = -i\phi_s / (1 + i\phi).
\]
Since \(y\) is a real valued function
\[
b(x, t) = -(1 + \phi_s)^{-1} \partial y / \partial x^{-1} \phi_t(y, s).
\]
Let the \(C^*\) map \(\psi(x, t) \rightarrow (y, s)\) be the local coordinate change as defined by (2) and \(r\) be the positive numbers such that
\[
\{(y, s) \mid |y-k| < \tilde{r}, |s| < \tilde{\delta} \} \subset \psi(\Omega)
\]
where \(k\) is a constant number as follows:

We first assume that \(L\) satisfies condition \((P_1)\). We claim \(\psi\) maps \(\{(0, t) \mid |t| < \tilde{\delta}\}\) into \(\{(k, s) \mid |s| < \tilde{\delta}\}\) in \((y, s)\) plane. In fact, from the condition \((P_1)\)
\[
b(0, t) = 0 \text{ for any } t, |t| < \tilde{\delta}.
\]
Therefore
\[
\partial / \partial t (\xi + i\eta) = 0 \text{ for any } t, |t| < \tilde{\delta}.
\]
So \(\xi(0, t) = k\) (a constant as above) for any \(t, |t| < \tilde{\delta}\).

From (3), \(\phi_s(k, s) = 0\) for any \(s, |s| < \tilde{\delta}\).

So \(\phi(k, s) = \alpha\) (a constant).

Since \(\psi\) is a bijective map, the inverse image of \(\{(y, s) \mid |y-k| < \tilde{r}, |s| < \tilde{\delta}\}\) for any \(k \neq k\) under \(\psi\) is entirely contained in \(\{(x, t) \in \Omega \mid x > 0\}\) or \(\{(x, t) \in \Omega \mid x < 0\}\).

Note that for any \(k \neq k\), \(|y-k| < \tilde{r}\), then the map \(s \rightarrow \psi(y, s)\) is a strictly increasing function in the interval \(|s| < \tilde{\delta}\).

Note that (i) of condition \((P_1)\) is a special case of four other kinds of signs in \(\{(x, t) \in \Omega \mid x > 0\}
\)
\(\cup \{(x, t) \in \Omega \mid x < 0\}\).

For instance, if \(b(x, t) < 0\) for any \((x, t) \in \Omega, x \neq 0\), then the map \(s \rightarrow \psi(y, s)\) is a strictly decreasing function for any \(k \neq k\), \(|y-k| < \tilde{r}\).

Now we subdivide the open rectangles
\[
|y-k| < \tilde{r}, |s| < \tilde{\delta}
\]
as a union of \(I = \{(k, s) \mid |s| < \tilde{\delta}\}\)
and a open rectangles \(R^+ = \{(y, s) \mid k < y < k + \tilde{r}, |s| < \tilde{\delta}\}\)
and \(R^- = \{(y, s) \mid k - \tilde{r} < y < k, |s| < \tilde{\delta}\}\).

We note that the ranges of the map \(z = y + i\phi(y, s)\) restricted to the rectangle (4) as follows:

(i) \(z\) maps \(I\) to the single point \(k + ia\)

(ii) \(z\) maps the rectangles \(R^+\) and \(R^-\) homeomorphically onto open sets \(\theta_1\) and \(\theta_2\) of the complex plane \(C\) which are entirely contained, respectively, in the strip \(k < Re\ z < k + \tilde{r}\) and in the strip \(k - \tilde{r} < Re\ z < k\).

We shall denote by \(A\) the image of the rectangle (4) under \(\psi\).

Let now \(f(x, t)\) be any \(C^*\) function in \(R^2\) with support contained in
\[
V = \psi^{-1}(\{(y, s) \mid |y-k| < \tilde{r}, |s| < \tilde{\delta}\}).
\]
We note the equation
\[ Lu = \partial u / \partial t + i b(x, t) \partial u / \partial x = f \] is equivalent to
\[ \left( \partial / \partial \tilde{s} + \lambda(y, s) \partial / \partial y \right) \left( u(y, s) - \int_{-s}^{s} f(y, \sigma) \, d\sigma \right) = -\lambda(y, s) \int_{-s}^{s} (\partial f / \partial y)(y, \sigma) \, d\sigma. \] \hspace{1cm} (5)
Here \( f(y, s) = f(x(y, s), t) \) etc.

For the simplicity we shall set
\[ v(y, s) = u(y, s) - \int_{-s}^{s} f(y, \sigma) \, d\sigma, \]
\[ g(y, s) = -\lambda(y, s) \int_{-s}^{s} (\partial f / \partial y)(y, \sigma) \, d\sigma. \]
\[ \lambda = -i \phi_y / 1 + i \phi \] vanishes identically on the vertical line segment \( I \) (where \( \phi = 0 \)).

Now we transform \( v \) and \( g \) to the set \( A \) under the map \( z = y + i \phi(y, s) \).

Since \( g = 0 \) on \( I \) and \( z \) is a homeomorphism on \( R^+ \) and \( R^- \), the transferred function \( g(z) \) can be extended by 0 outside of \( A \) and is equal to compactly supported function of \( L^1 \) class, with a compact support contained in \( \tilde{A} \).

The equation (5) becomes
\[ \left( \partial \tilde{z} / \partial \tilde{s} + \lambda(y, s) \partial \tilde{z} / \partial y \right) \left( \partial b / \partial \tilde{z} \right) = \tilde{g}, \]
where \( v \) denotes \( v(y, s) \) as a function of \( z \).

But since \( \lambda(y, s) \partial \tilde{z} / \partial s = -\partial \tilde{z} / \partial \tilde{s} \), we have \( \partial \tilde{z} / \partial s = -\lambda \partial \tilde{z} / \partial y \).

Therefore (5) reads to
\[ 2i \left( \text{Im} \lambda \right) \left( \partial \tilde{z} / \partial y \right) \left( \partial b / \partial \tilde{z} \right) = \tilde{g}. \] \hspace{1cm} (6)

Moreover, since \( \partial \tilde{z} / \partial y = 1 - i \phi \) and \( \text{Im} \lambda = -\phi / 1 + i \phi \), (6) equivalent to
\[ \left( -2i / 1 + i \phi \right) \phi = \tilde{g} \] \hspace{1cm} (7)
or
\[ \partial \tilde{b} / \partial \tilde{z} = i / 2 \left[ (1 + i \phi) \phi \right]^{-1} \left[ -1 / 2 \int_{-s}^{s} f(y, \sigma) \, d\sigma \right]. \] \hspace{1cm} (8)

(8) is a inhomogeneous Cauchy–Riemann equation whose solution is given by
\[ v = 1 / 2 \pi i \int \left[ F(\phi) / z - \phi \right] \, d\phi / dz \]
where
\[ F(z) = i / 2 \left[ (1 + i \phi) \phi \right] \]
\[ = \left[ -1 / 2 \int_{-s}^{s} f(y, \sigma) \, d\sigma \right]^{-1}. \] \hspace{1cm} (9)

To revert (9) to \( y, s \) coordinates, we set
\[ \phi = y' + i \phi(y', s'). \]

Then we have
\[ d\phi / dz = 2i \phi', \]
and hence
\[ v(y, s) = 1 / 2 \pi \int_{y'} \phi' \left( y', s' \right) (y - y' + i \phi(y, s) - \phi(y', s')) dy' / dz'. \] \hspace{1cm} (10)

where
\[ k(y, s) = \int_{-s}^{s} (\partial f / \partial y)(y, \sigma) \, d\sigma. \]

Since \( v(y, s) \) is the pullback via \( (y, s) \mapsto y + i \phi(y, s) \) of \( \tilde{v} \) which is locally \( L^1 \) function, \( v(y, s) \) is
well defined and in fact, a $C^\infty$ solution.

Then for any $f \in C_0^\infty(V)$ a $C^\infty$ solution of $Lu = f$ in $V$ is given by the pullback via the map $\phi: (x, t) \longrightarrow (y, s)$ defined in (2) of a $C^\infty$ solution

$$u(y, s) = -1/2\pi \iint_{R_1} \phi_s(y', s') k(y', s')/\left((y - y + i(\phi(y, s) - \phi(y', s'))\right) dy' ds'
+ \int_{-1}^s f(y, \sigma) d\sigma$$

where

$$k(y, s) = \int_{-1}^s (\partial f/\partial y)(y, \sigma) d\sigma.$$

So far we considered only the case when $L$ satisfies $(P_1)$. When $L$ satisfies $(P_2)$, the argument is much simpler, as $\varepsilon$ is a homeomorphism on the entire rectangle $(4)$ in this case.

References