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Probability Measure and the Daniell Integral

by Hoo Jin O

Kongju National Teachers College, Kongju, Korea

1. Introduction

It is sometimes convenient to introduce integration directly without using the concept of measure. This happens when we have an elementary integral I defined on some class L of elementary functions. Our purpose of this paper is to describe the Daniell integral and to show its connection.

In this paper we shall define the Daniell integral and probability measure. And we showed that there exists a unique probability measure on σ-field of vector lattice VL and investigated some theorems about measures on $\sigma(VL)$.

2. Definitions and Remarks

Definition 1. Let VL be a vector lattice. Then a positive linear functional I on VL is called a *Daniell integral* if the following condition is satisfied.

If (f_n) is a nonincreasing sequence in VL converging to zero, then $I(f_n)$ converges to zero.

Theorem 1. *Let VLU be the collection of all extended real-valued functions on X of the form $\sup f_n$, where (f_n) is a nondecreasing sequence of nonnegative functions in VL. Let $J=\{G\subset X : \chi_G\in VLU\}$ and define $(G)=I(\chi_G)$, where I is the extension of I to VLU. Then the VL-open sets and the extended nonnegative function μ on these sets satisfy the following conditions:*

(*i*) *If $G_1, G_2\in J$, then $G_1\cup G_2$, $G_1\cap G_2\in J$ and*

$$\mu(G_1\cup G_2)+\mu(G_1\cap G_2)=\mu(G_1)+\mu(G_2).$$

(*ii*) *If $G_1, G_2\in J$ and $G_1\subset G_2$, then $\mu(G_1)\leqslant\mu(G_2)$.*

(*iii*) *If $G_n\in J$, $n=1,2,\cdots$ and $G_n\uparrow G$, then $G\in J$ and $\mu(G_n)\uparrow\mu(G)$.*

Proof. (i) By the definition of

$$\begin{aligned}\mu(G_1\cup G_2)+\mu(G_1\cap G_2)&=I(\chi_{G_1\cup G_2})+I(\chi_{G_1\cap G_2})\\&=I(\chi_{G_1})+I(\chi_{G_2})\\&=\mu(G_1)+\mu(G_2).\end{aligned}$$

(ii) This follows from the definition of μ.

(iii) Let (f_n) be a sequence in VLU and $f_n\uparrow f$. Then $f\in VLU$ and $I(f_n)\uparrow I(f)$. Thus $\mu(G_n)=I(\chi_{G_n})\uparrow I(\chi_G)=\mu(G)$.

Definition 2. A measure on a σ-field F of subsets of X is a nonnegative, extended real-valued function μ on F such that

$$\mu(\cup A_n)=\sum\mu(A_n)$$

for every sequence A_n of pairwise disjoint sets of F.

If $\mu(X)=1$, μ is called a *probability measure.*

Theorem 2. *Let* $J=\{G\subset X : \chi_G\in VLU\}$ *and define* $\mu(G)=I(\chi_G)$, $G\in J$. *If we assume that all constant functions belong to VL and* $I(1)=1$ *(hence* $I(c)=c$ *for all c), then J and* μ *satisfy the following conditions:*

(*i*) *If* $\phi, X\in J$, *then* $\mu(\phi)=0$, $\mu(X)=1$, $0\leqslant\mu(A)\leqslant 1$ *for all* $A\in J$.

(*ii*) *If* $G_1, G_2\in J$, *then* $G_1\cup G_2$, $G_1\cap G_2\in J$ *and* $\mu(G_1\cup G_2)+\mu(G_1\cap G_2)=\mu(G_1)+\mu(G_2)$.

(*iii*) *If* $G_1, G_2\in J$ *and* $G_1\subset G_2$, *then* $\mu(G_1)\leqslant\mu(G_2)$.

(*iv*) *If* $G_n\in J$, $n=1,2,\cdots$ *and* $G_n\uparrow G$, *then* $G\in J$ *and* $\mu(G_n)\uparrow\mu(G)$.

Proof. (i) Since VL contains the constant functions and $I(c)=c$, condition (i) holds.

(ii), (iii) and (iv) These statements follow from theorem 1.

Theorem 3. *Under the hypothesis of theorem 2, let* $Y=\{H\subset X : \mu^*(H)+\mu^*(H^c)=1\}$ *and define*

$$\mu^*(A)=inf\{\mu(G) : G\in J,\ G\supset A\}. \tag{1}$$

Then μ^* *is a probability measure on the σ-field Y and* $\mu^*=\mu$ *on J.*

Proof. By the definition of μ^* we have $\mu^*=\mu$ on J.

Since $\mu^*(A\cup B)+\mu^*(A\cap B)\leqslant\mu^*(A)+\mu^*(B)$, for $H_1, H_2\in X$,

$$\mu^*(H_1\cup H_2)+\mu^*(H_1\cap H_2)\leqslant\mu^*(H_1)+\mu^*(H_2), \tag{2}$$

and since $(H_1\cup H_2)^c=H_1^c\cap H_2^c$, $(H_1\cap H_2)^c=H_1^c\cup H_2^c$,

$$\mu^*(H_1\cup H_2)^c+\mu^*(H_1\cap H_2)^c\leqslant\mu^*(H_1^c)+\mu^*(H_2^c). \tag{3}$$

Adding (2) and (3), we have

$$\mu^*(H_1\cup H_2)+\mu^*(H_1\cup H_2)^c=1 \tag{4}$$

and

$$\mu^*(H_1\cap H_2)+\mu^*(H_1\cap H_2)^c=1. \tag{5}$$

Hence $H_1\cup H_2$, $H_1\cap H_2\in Y$. Thus Y is a field.

By (4)and (5) we have $\mu^*(H_1\cup H_2)+\mu^*(H_1\cap H_2)=\mu^*(H_1)+\mu^*(H_2)$.

Therefore μ^* is finitely additive on Y.

Let $H_n\in Y$, $n=1,2,\cdots, H_n\uparrow H$; $u^*(H)+\mu^*(H^c)\geqslant 1$.

But $\mu^*(H)=lim_{n\to\infty}\mu^*(H_n)$, hence for any $\varepsilon>0$, $\mu^*(H)\leqslant\mu^*(H_n)+\varepsilon$ for large n. Since $\mu^*(H^c)\leqslant\mu^*(H_n^c)$ for all n and $H_n\in Y$, we have $\mu^*(H)+\mu^*(H^c)\leqslant 1+\varepsilon$. Since ε is arbitrary, $H\in Y$. Hence Y is a σ-field. Since $\mu^*(H_n)\to\mu^*(H)$, μ^* is countably additive. Thus μ^* is a probability measure on Y.

Theorem 4. *Let VL be a vector lattice of functions on a set X and let I be a Daniell integral on VL. Then there is a unique measure* ν *on* $\sigma(VL)$ *such that*

$$I(f)=\int f d\nu \text{ for all } f \text{ in } VL,$$

and

$$\nu(A)=inf\{\mu(G) : G\supset A \text{ and } G \text{ is } VL\text{-open}\}. \tag{6}$$

Proof. Let ν be the restriction of μ^* defined by equation (1) to $\sigma(VL)$. Obviously equation (6) is satisfied. Suppose G is VL-open. Then

$$I(\chi_G)=\mu(G)=\mu^*(G)=\nu(G)=\int_X \chi_G d\nu. \tag{7}$$

If $f\in VL$ and $f\geqslant 0$ then $f=sup\ h_n$, where

$$h_n = \frac{1}{2^n} \sum_{k=1}^{n2^n} \chi_{\{k/2^n < f\}}.$$

Each $\{k/2^n < f\}$ is VL-open so that h_n is in VLU. Using equation (3), we have

$$I(h_n) = \frac{1}{2^n} \sum_{k=1}^{n2^n} I(\chi_{\{k/2^n < f\}}) = \frac{1}{2^n} \sum_{k=1}^{n2^n} \int_X \chi_{\{k/2^n < f\}} d\nu = \frac{1}{2^n} \int_X \sum_{k=1}^{n2^n} \chi_{\{k/2^n < f\}} d\nu = \int_X h_n d\nu.$$

Since $I(h_n) \uparrow I(f)$, we have

$$I(f) = lim \int_X h_n d\nu = \int_X f d\nu.$$

Also since $0 \leqslant I(f) < \infty$, f is ν-integrable. If f is an arbitrary function in VL the integrability of f and the equation $I(f) = \int_X f d\nu$ follows by writing f as the difference of the nonnegative functions f^+ and f^- in VL.

For any VL-open set G, there exists a sequence (f_n) from VL with $f_n \geqslant 0$ and $f_n \uparrow \chi_G$. Therefore if ν' is any measure such that $I(f) = \int_X f d\nu'$ for all f in VL, then

$$\nu'(G) = \int \chi_G d\nu' = \lim_n \int f_n d\nu' = \lim_n I(f_n) = \mu^*(G).$$

Since $\nu'(G) = \mu^*(G)$ for each VL-open set G, $\nu = \nu'$ from equation (7).

Theorem 5. *Under the hypothesis of theorem* 4, *assume that* $I(1) = 1$. *Then there is a unique probability measure* P *on* $\sigma(VL)$ *such that each* $f \in VL$ *is* P*-integrable and* $I(f) = \int_X f dp$.

Proof. Let P be the restriction of μ^* to $\sigma(VL)$. Since $\sigma(VL) = \sigma(J)$, P is a probability measure by theorem 3. If $G \in J$, then

$$I(\chi_G) = \mu(G) = \mu^*(G) = P(G) = \int_X \chi_G dp.$$

The existence of the desired probability measure P follows from the preceding theorem. If P' is another such measure, then $\int_X f dp = \int_X f dp'$ for all $f \in VL$, and hence for all $f \in VLU$. By setting $f = \chi_G$, $G \in J$, we have $P = P'$ on J. Since J is closed under finite intersection by theorem 1, $P = P'$ on J.

요 약

본 논문에서는 Daniell 적분과 확률 측도를 정의하고, 이와 연관된 성질들을 이용하여 벡터속으로 이루어진 σ체 위에 정의된 확률측도는 유일하게 존재함을 밝혔다.

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