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On the Serret-Frenet Equations

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1. Introduction

It is well known that along a space curve $x^i=x^i(s)$* of class 3-the vectors $\boldsymbol{t}(\alpha^i)$, $\boldsymbol{n}(\beta^i)$ and $\boldsymbol{b}(\gamma^i)$ satisfy

$$\frac{d\alpha^i}{ds}=\kappa\beta^i,$$
$$\frac{d\beta^i}{ds}=-(\kappa\alpha^i+\tau\gamma^i), \tag{1.1}$$
$$\frac{d\gamma^i}{ds}=\tau\beta^i,$$

in the cartesian coordinates x^i. They take a central position in the theory of space curves and are known as the formulas of Frenet, or of Serret-Frenet.

The purpose of the present paper is to introduce the concept of coordinate transformation and some preliminary results and to obtain the generalized tensorial form of the Frenet formulas in the general coordinate system.

No new results are derived in this paper, but the method employed here is more simple to whom uses a little knowledge of tensor analysis.

2. Preliminary Results

A set of 3-functions $f^i(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, whose functional determinant is not equal to zero,

$$x^i=f^i(\bar{x}_1, \bar{x}_2, \bar{x}_3), \quad i=1, 2, 3, \tag{2.1}$$

defines a transformation of coordinates. The system (2.1) can be solved for $\bar{x}$'s in terms of x's giving

$$\bar{x}^i=g^i(x^1, x^2, x^3). \tag{2.2}$$

The equations (2.1) and (2.2) enable us to pass from either system of coordinates to the other.

Riemann defined the infinitesimal distance ds between the adjacent points, whose coordinates in any system are x^i and x^i+dx^i, by

$$ds^2=g_{ij}dx^idx^j, \tag{2.3}$$

where the coefficients g_{ij}, called Riemannian metric tensor, are functions of the coordinates x^i. There is no loss of generality in assuming g_{ij} the symetric covariant tensor of order 2.

The positive definiteness of the form (2.3) implies that

$$g \overset{\text{def.}}{=} |g_{ij}| \neq 0. \tag{2.4}$$

Its reciprocal tensor g^{ij}, defined by

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* x's are 3 dimensional cartesian coordinates and throughout this paper Roman indices run from 1 to 3 and follow the summation convention.

$$g_{ik}g^{ij}=\delta_{kj}, \tag{2.5}$$

is called the fundamental contravariant tensor.

The Christoffel symbols $[k, ij]$ and $\left\{ {k \atop i\ j} \right\}$ of the first and second kind respectively are used to denote the functions

$$[k,\ ij] \overset{\text{def.}}{=} \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right), \tag{2.6}$$

$$\left\{ {k \atop i\ j} \right\} \overset{\text{def.}}{=} g^{kh}[h,\ ij].$$

Note. In cartesian coordinates, $[k,\ ij]=\left\{ {k \atop i\ j} \right\}=0.$

We can express the second derivatives of x's with respect to the $\bar{x}$'s in terms of their first derivatives and the Christofell symbols for both systems:

$$\overline{\left\{ {l \atop i\ j} \right\}}\frac{\partial x^d}{\partial \bar{x}^l}=\left\{ {d \atop a\ b} \right\}\frac{\partial x^a}{\partial \bar{x}^i}\ \frac{\partial x^b}{\partial \bar{x}^j}+\frac{\partial^2 x^d}{\partial \bar{x}^i \partial \bar{x}^j}. \tag{2.7}$$

We consider now a space curve with $\bar{x}^i=\bar{x}^i(t)$. Using (2.2), we have equations of the curve in terms of x^i and t. The derivatives $\dfrac{dx^i}{dt}$ are given by

$$\frac{dx^i}{dt}=\frac{\partial x^i}{\partial \bar{x}^j}\ \frac{d\bar{x}^j}{dt}. \tag{2.8}$$

Differentiating (2.8) with respect to t, we have

$$\frac{d^2x^i}{dt^2}=\frac{\partial^2 x^i}{\partial \bar{x}^j \partial \bar{x}^k}\ \frac{d\bar{x}^j}{dt}\ \frac{d\bar{x}^k}{dt}+\frac{\partial x^i}{\partial \bar{x}^j}\ \frac{d^2\bar{x}^j}{dt^2}, \tag{2.9}$$

which in consequence of (2.7) may be written in the form

$$\frac{d^2x^i}{dt^2}+\left\{ {i \atop h\ l} \right\}\frac{dx^h}{dt}\ \frac{dx^l}{dt}=\left(\frac{d^2\bar{x}^j}{dt^2}+\overline{\left\{ {j \atop a\ b} \right\}}\frac{d\bar{x}^a}{dt}\ \frac{dx^b}{dt}\right)\frac{\partial x^i}{\partial \bar{x}^j}. \tag{2.10}$$

3. Main Theorem

Theorem (Serret-Frenet formulas). *When a curve C is defined in any system of the coordinates x^i, the Frenet formulas are*

$$\begin{aligned} \frac{d\alpha^i}{ds}+\left\{ {i \atop j\ k} \right\}\alpha^j\frac{dx^k}{ds}&=\kappa\beta^i, \\ \frac{d\beta^i}{ds}+\left\{ {i \atop j\ k} \right\}\beta^j\frac{dx^k}{ds}&=-(\kappa\alpha^i+\tau\gamma^i), \\ \frac{d\gamma^i}{ds}+\left\{ {i \atop j\ k} \right\}\gamma^i\frac{dx^k}{ds}&=\tau\beta^i, \end{aligned} \tag{3.1}$$

where α^i, β^i and γ^i are the contravariant components of the unit tangent, unit principal normal and unit binormal respectively of C, s being the arc length of C, and κ and τ the curvature and torsion respectively.

Proof. If the arc s is the parameter we have

$$\alpha^i=\frac{dx^i}{ds}=\frac{\partial x^i}{\partial \bar{x}^j}\bar{\alpha}^j,$$

$$\beta^i=\frac{\partial x^i}{\partial \bar{x}^j}\bar{\beta}^j, \tag{3.2}$$

$$\gamma^i = \frac{\partial x^i}{\partial \bar{x}^j} \bar{\gamma}^j.$$

In cartesian coordinates x^i, we have from (1. 1)

$$\frac{d^2 \bar{x}^j}{ds^2} = \kappa \bar{\beta}^j, \tag{3. 3}$$

where κ is the curvature of the curve C.

In any coordinates x^i in consequence of (3. 2) and (3. 3) we have from (2. 10)

$$\frac{d^2 x^i}{ds^2} + \left\{ {i \atop h\ l} \right\} \frac{dx^h}{ds} \frac{dx^l}{ds} = \kappa \beta^i,$$

that is,

$$\frac{d\alpha^i}{ds} + \left\{ {i \atop h\ l} \right\} \frac{dx^h}{ds} \frac{dx^l}{ds} = \kappa \beta^i. \tag{3. 4}$$

If we differentiate (3. 2) with respect to s, and make use of (1. 1) (2. 7) and (3. 2), we obtain

$$\frac{d\beta^i}{ds} = -(\kappa \bar{\alpha}^j + \tau \bar{\gamma}^j) \frac{\partial x^i}{\partial \bar{x}^j} + \bar{\beta}^j \frac{\partial^2 x^i}{\partial \bar{x}^j \partial \bar{x}^k} \frac{d\bar{x}^k}{ds}$$

$$= -(\kappa \bar{\alpha}^j + \tau \bar{\gamma}^j) \frac{\partial x^i}{\partial \bar{x}^j} - \bar{\beta}^j \left\{ {i \atop h\ l} \right\} \frac{\partial x^h}{\partial \bar{x}^j} \frac{\partial x^l}{\partial \bar{x}^k} \frac{d\bar{x}^k}{ds}$$

$$= -(\kappa \alpha^i + \tau \gamma^i) - \left\{ {i \atop h\ l} \right\} \beta^h \frac{\partial x^l}{ds},$$

$$\frac{d\gamma^i}{ds} = \tau \bar{\beta}^j \frac{\partial x^i}{\partial \bar{x}^j} - \left\{ {i \atop h\ l} \right\} \bar{\gamma}^j \frac{\partial x^h}{\partial \bar{x}^j} \frac{\partial x^l}{\partial \bar{x}^k} \frac{\partial \bar{x}^k}{ds}$$

$$= \tau \beta^i - \left\{ {i \atop h\ l} \right\} \gamma^h \frac{\partial x^l}{ds}.$$

These results and (3. 4) prove the theorem.

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