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—Research Paper—

A Note on the Optimum Character of One-Sided Sequential Probability Ratio Tests

Volker Abel*

Abstract

We Observe a sequence of *i. i. d* random variables with density f or g. Only if g is true we should stop the process. Hence. the testing problem is completely described by a stopping time. Among all stopping times with error probability of first kind not exceeding a given bound, the one-sided sequential probability ratio test has smallest expected sample size if g is true. Moreover, the generalized one-sided SPRT has smallest expected sample size for g in the class of stopping times with expected sample size under f not falling below a given bound.

1. INTRODUCTION

Let $y_1, y_2, \cdots$ be independent and identically distributed random variables observable one at a time and having probability density f or g with respect to a sigma-finite measure μ on the space where each y takes values. We assume that f and g are non-equivalent. Testing whether f or g is the true density the sequential probability ratio test (SPRT) is best in the following sense. Among all (fixed-sample or sequential) tests whose error probabilities do not exceed those of the SPRT, the latter has the smallest expected sample size for both f and g. This statement was first proved, for the subclass of tests with finite expected sample size for both f and g, by Wald and Wolfowitz [10]. Since then, the statement as well as its proof has undergone some refinements. For an account of the literature and a new, elegant derivation of the SPRT's optimal property from a more general setting, the so-called modified Kiefer-Weiss problem, see the article of Lorden[6].

While the SPRT is best in the broad class of tests controlling both sample size and terminal decision, it is also interesting to have best tests in the smaller class with sample size or terminal decision not at the stattistician's disposal. For fixed sample sizes, Neyman and Pearson [7] gave in 1933 the best test, i.e. the most powerful one for a given level of significance. The present paper is concerned with the contrary extreme. We confine ourselves to tests which

* Department of Computer Science, University of the Federal Armed Forces Munich, Neubiberg, Federal Republic of Germany

are pure stopping times, since the terminal decision always is, say g. Such a situation is typical for the area of statistical process control. If the process is in control (i.e., f is true) we would like to continue sampling indefinitely, thus not interrupting the process. In the case of the process being out of control(i.e., g is true), sampling should be stopped as soon as possible. By heavily drawing on chapter 5 of Chow, Robbins and Siegmund [1], we will show that one-sided SPRTs are 'optimal' for this testing problem. (A one-sided SPRT is a SPRT without lower boundary.) Since there are two ways of formalizing the desire to continue sampling when f is true, we will have two optimality criteria, one leading to ordinary one-sided SPRTs with fixed decision boundary, the other one to generalized one-sided SPRTs with moving boundary.

In section 2, the optimality properties are stated, followed by their proofs in section 3. Some remarks concerning previous research on the subject matter, particularly on power-one tests, are given in section 4.

2. OPTIMALITY PROPERTY OF THE ONE-SIDED SPRT

The tests we have to consider are extended stopping times N, $0 \leq N \leq \infty$, defined on a proper product space, with respect to a sequence of sigma-fields $F_0 \subset F_2 \subset \cdots\cdots$. (We partly join notation and terminology of [6].)

Let F and G be the probabilities referring to f and g, and let f_n and g_n denote the likelihoods (Radon-Nikodym derivatives) for $n=0,\ 1, \cdots$ observations, e.g.

$$f_n = f(y_1)\cdots f(y_n) \text{ for } n \geqq 1,\ f_0 \equiv 1.$$

Definition. A one-sided sequential probability ratio test is a stopping time S such that for some $A>0$ and for $n=0, 1, \cdots$

$$\{S=n\} \subset \{g_n/f_n \geqq A\}$$

$$\{S>n\} \subset \{g_n/f_n \leqq A\} \quad a.s.\ F,G.$$

A generalized one-sided sequential probability ratio test is a stopping time T such that for $n=0, 1, \cdots$ and some $A_n>0$

$$\{T=n\} \subset \{g_n/f_n \geqq A_n\}$$

$$\{T>n\} \subset \{g_n/f_n \leqq A_n\} \quad a.s.\ F,G.$$

As indicated in the section above there are two ways to describe the desire to continue sampling indefinitely if f is true. We may control the probability of stopping if f is true, $F(N<\infty)$, i.e., doing the error of first kind. Alternatively, we may require the expected stopping time $E_F(N)$ to exceed a given bound. Hence we have two different conditions when looking for a stopping time which minimizes the expected stopping time $E_G(N)$.

Let $0<\alpha<1$ and $\gamma>0$.

Proposition 1. Let S be a one-sided SPRT with $F(S<\infty)=\alpha$. If N is a stopping time with $F(N<\infty) \leqq \alpha$, then $E_G(N) \geq E_G(S)$.

Proposition 2. Let T bd a generalized one-sided SPRT with $E_F(T)=\gamma$. If N is a stopping time with $E_F(N) \geqq \gamma$, then $E_G(N) \geqq E_G(T)$.

3. PROOFS

Unlike the proofs of the SPRT's optimum character, which all solve an auxiliary Bayesian problem, we can directly approach the proofs of propositions 1 and 2 by using the method of undetermined multipliers.

Lemma. Let u and v be real-valued functions defined over a space Z.

a) If $z_0 \in Z$ minimizes $u(z)+k\cdot v(z)$ for some $k>0$, and if $v(z_0)=c$, then z_0 minimizes $u(z)$ in the set $\{z\in Z: v(z)\leqq c\}$.

b) If $z_0 \in Z$ minimizes $u(z)-\lambda\cdot v(z)$ for some $\lambda>0$, and if $v(z_0)=c$, then z_0 minimizes $u(z)$ in the set $\{z\in Z: v(z)\geq c\}$.

The lemma is easy to prove and of some utility in statistical hypotheses testing (see e.g. [4]). With Z being the set of stopping times, u the expected stopping time under G and v the probability of finite termination, and the expected stopping time under F, respectively, this lemma allows to prove propositions 1 and 2 by means of methods given in chapter 5 of [1].

Proof of proposition 1. It is sufficient to minimize

$$E_G(N)+k\cdot F(N<\infty).$$

This expression equals

$$\sum_{n=0}^{\infty}\int_{\{N=n\}} n\cdot g_n d\mu^n + k\cdot\sum_{n=0}^{\infty}\int_{\{N=n\}} f_n d\mu^n$$

$$=\sum_{n=0}^{\infty}\int_{\{N=n\}} (n+k\cdot f_n/g_n)\cdot g_n d\mu^n$$

$$=E_G(N+k\cdot f_N/g_N).$$

We set $-x_n=n+k\cdot f_n/g_n$ for $n=0,1,\cdots$, and can immediately apply the corollary of [1, p. 105]. Observing that f_n/g_n is a stationary Markov sequence, and defining $B=\{z\in IR^+ : -k\cdot z\geqq V_0(z)\}$, where $V_0(z)$ is the 'value' of the stochastic sequence $\{-n-k\cdot f_n/g_n,\ F_n\}_0^\infty$ as defined in [1], we get as optimal stopping time σ:

$$\sigma=\inf\{n\geqq 1 : f_n/g_n\in B\}.$$

Now, $V_0(\cdot)$ is continuous and convex, and $V_0(0)=1$. Thus, there is a constant $c>0$, which is a function of k, such that $B=\{z\in IR^+ : z\leqq c\}$, and

$$\sigma=\inf\{n\geqq 1 : g_n/f_n\geqq c^{-1}\}.$$

Noting that σ is a one-sided SPRT and taking a one-sided SPRT S with $F(S<\infty)=\alpha$, proposition 1 is proved.

Proof of proposition 2. We minimize

$$E_G(N)-\lambda\cdot E_F(N)$$

$$=\sum_{n=0}^{\infty}\int_{\{N=n\}} (n-\lambda\cdot n\cdot f_n/g_n)g_n d\mu^n$$

$$= E_G(N - \lambda \cdot N \cdot f_N / g_N).$$

now, for $n=0, 1, \cdots$, we put $x_n = \lambda \cdot n \cdot f_n / g_n - n$, $z_n = f_n / g_n$, $w_n(z_n) = -1$, and $\varphi_n(z_n) = \lambda \cdot n \cdot z_n \cdot$ By means of the remark in [1, p. 105], we then get as optimal stopping time

$$\inf\{n \geqq 1 : \varphi_n(z_n) \geqq V_n(z_n)\},$$

where $V_n(z_n)$ is again as in (1). Since $\varphi_n(0)=0$, $V_n(0) = -n$, and $V_n(\cdot)$ convex, there exist $c_n > 0$, $n=0, 1, \cdots$, depending on λ, such that

$$\{z_n : \varphi_n(z_n) \geqq V_n(z_n)\} = \{z_n : z_n \leqq c_n\}.$$

Thus, the optimal stopping time is

$$\inf\{n \geqq 1 : g_n / f_n \geqq c_n^{-1}\},$$

which is a generalized one-sided SPRT. Taking a generalized one-sided SPRT T with $E_F(T) = \gamma$ completes the proof.

4. SOME REMARKS

The optimality property of one-sided SPRTs, as stated in proposition 1, was first mentioned, without proof, for normal densities by Darling and Robbids [2]. Later, the optimality property was claimed for densities of a one-parameter family by Robbins and Siegmund [9], referring to [1] for a proof. Indeed, from a purely technical point of view, this indication is suitable, and our proof is nearly identical to [1, p. 108]. However, in [1], a linear combination of $F(N<\infty)$ and $E_G(N)$ is minimized with a priori probabilities π for f and $1-\pi$ for g as weights. Even putting $k = \pi/(1-\pi)$ does not support the interpretation as in [9], since k is then determined by π and can not be chosen appropriately as required in our proof.

If the densities in question belong to the one-parameter exponential family the likelihood ratio of the (generalized) one-sided SPRT can be expressed through the sample sum. The resulting stopping time is often called 'power-one test' or 'open-ended test' (e.g. Lai [3] and Lorden [5]). Though the literature about these tests has grown considerably in the last decade, to our knowledge it has not previously made clear if and in which sense they are optimal.

As for two-sided SRRTs it is extremely difficult, too, to determine the boundaries A and A_n of one-sided SPRTs. In the one-parameter exponential case we may use the literature on power-one tests where approximations of average sample sizes can be found (e.g. [3, 8]).

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REFERENCES

1. Chow, Y. S., Robbins, H., Siegmund, D.: Great Expectations: The Theory of Optimal Stopping. Boston: Houghton Mifflin 1971.

3. Lai, T.L.: Power-onetestsbasedon sample sums. Ann. Statist. 5, 866~880(1977).
4. Lehmann, E.: Testing Statistical Hypotheses. New York: Wiley 1959.
5. Lorden, G.: Open-ended tests for Koopman-Darmois families. Ann. Statist. 1, 633~643(19 73).
6. Lorden, G.: Structure of sequential tests minimizing an expected sample size. Z. Wahrs-
2. Darling, D.A., Robbins, H.: Some further remarks on Inequalities for sample sums. Proc. Nat. Acad. Sci. U.S.A. 60, 1175~1182(1968).
cheinlichkeitstheorie verw. Gebiete 51, 291~302(1980).
7. Neyman, J., Pearson, E.S.: On the problem of the most efficient tests of statistical hypotheses. Philos. Trans. Roy. Soc. Ser. A. 231, 289~337(1933).
8. Pollak, M., Siegmund, D.: Approximations to the expected sample size of certain sequential tests. Ann. Statist. 3, 1267~1282(1975).
9. Robbins, H., Siegmund, D: The expected sample size of some tests of power one. Ann, Statist. 2, 415~436(1974).
10. Wald, A, Wolfowitz, J.: Optimum character of the sequential probability ratio test. Ann. Math. Statist. 19, 326~339(1948).