The Muslim Mathematics

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Muslim의 수학

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=요 약=

Muslim의 수학은 천문학과 마찬가지로 종교적 필요성이 있었다. 기하학적 지식은 매일 예배하는 Mecca의 방향을 정하기 위한 것이었다면, 산술과 대수는 주로 祭日을 계산하기 위해 필요했다. Muslim의 수학은 결국 종교적 욕구를 충족시키는데 끝없을 뿐 과학 일반에의 응용은 없었으나, 전 인류에게 그 지식을 보급시킴으로서 수학사상 중요한 위치를 차지한다.

Muslim의 문화가 일어나는 시대는 10세기로 생각할 수 있으며, 특히 이 시기에 학술연구가 시작됐다. 11세기는 Muslim의 황금시기이며 실험과 이론의 두분야에서 눈부신 발달이 있었다. 12세기에는 退落의 과정을 밟으면서도 그 문화적 유산은 서구세계에 넘겼다. 본 논문에서는 산술, 대수, 삼각법을 중심으로 이들 학문의 형성과정과 지식의 전달과정을 살핀다.

I. ARITHMETIC

Mathematics, the man-made universe, appears to have emerged from man's primitive needs to keep records, to communicate information, and to undershand and control his environment. Certainly, arithmetic was among the first branches of mathematics to develop and flourish as the concepts of number and operations on numbers came into general usage. This development was no doubt a gradual one, but the advantages of counting soon led to the improvement and extension of basic mathematical concepts, which have proliferated over the centuries into what we today refer to as number theory.

It is believed that arithmetic came into existence before written language developed. Thus, the history of mathematics, stemming from arithmetic, is a part of the history of civilization. Moreover, the progress of man over the era of recorded history is largely paced by his use and grasp of mathematical ideas. The use and manipulation of symbols as mental repesentations of physical things led to the idea of the early use of mathematical operations of addition and subtraction without the need to count the real objects in a set.[1]

Arithmetic is the foundation of all mathematics, pure or applied. It is the most useful of all sciences, and there is, probably, no other branch of human knowledge which is more widely spread among the masses.[2]

Among the Muslim mathematicians who contributed the most to arithmetic was Abu-Yusef Ya'qub ibn Ishaq Al-Kindi.[3] Al-Kindi was born about 801 AD at Kufah during the governorship of his father. This position was occupied earlier by his grandfather. The surname indicates ancestry in the royal tribe of Kindah of Yamanite origin. Al-Kindi is known in the West as Alkindus.[4] To his people he became known as Faylusaf Al-Arab,[5] the philosopher of the Arabs. He was the only notable philosopher of pure Arabian blood and the first one in Islam. Al-Kindi 'was the most learned of his age, unique among his contemporaries in the knowledge of the totality of ancient sciences, embracing logic, philosophy, geometry, mathematics, music, and astrology;[6] According to Professor Emeritus Philip K. Hitti of Princeton University:

He [Al-Kindi] was a man with a first-class mind which addressed itself to the study of the new philosophy. It was an encyclopaedic mind to which no aspect of human knowledge seemed alien.[7]

Among his contributions to arithmetic, Al-Kindi wrote eleven texts on the subject. The following is a composite listing of the titles of these works:

1. An Introduction to Arthmetic

1) Iohn Desmond Bernal, *Sciinca in History* (*London, C. A.* Watts and Company, 1957), p.79.
2) Robias Dantzig, *Number, The Languge of Science* (Garden City, New York, Doubleday and Company, 1956), p. 38.
3) David Eugene Smith and Louis Charles Karpinski, *The Hindu-Arabic Numerals* (Boston, Ginn and Company, 1911), p. 10.
4) Ibn Al-Nadim, *Al-Fahrasat Li Ibn Al-Nadim* (Cairo, AlHaj Mustafa Muhammed, 1800), pp. 371-2.
5) Yuhana Gamir, *Falasifat Al-Arab* (Beirut, Al-Muktabat Ash Shargiyah, 1957), p. 5.
6) Franklin Wesley Kokomoor, *Mathematics in Human Affairs*(New York, Prentice-Hall, 1945), p. 172.
7) Philip K. Hitti, *Makers of Arab History* (New York, Harper and Row Publishers, 1968), p. 187.

2. Manuscipt on the Use of Indian Numbers
3. Manuscript on Explanation of the Numbers mentioned by Plato in his politics
4. Manuscript on the Harmony of Numbers
5. Manuscript of Unity from the Point of View of Numbers
6. Manuscript on Elucidating the Implied Numbers
7. Manuscript on Prediction from the Point of View of Numbers
8. Manuscript on Lines and Multiplication with Numbers
9. Manuscript on Relative Quantity
10. Manuscript on Measuring of Proportions and Times
11. Manuscript on Numerical Procedures and Cancellation[8]

Al-Karkhi of Baghdad (1020 AD) was the most scholarly and the most original writer of arithmetic. Two of his works are known. The first is the Al-Kafi fi al-Hisab (Essentials of Arithmetic), which gives the rules of computations. His second work, Al-Fakhri, derives its name from Al-Karkhi's friend, the grand vizier in Baghdad at that time.[9]

A Latin translation of a Muslim arithmetic text was discovered in 1857 at the Library of the University of Cambridge. Entitled Algoritmi de numero Indorum, the work opens with the words: 'Spoken has Algoritmi. Let us give deserved praise to God, our leader and defender.'[10] It is believed that this is a copy of Al-Khwarizmi's arithmetic text which was translated into Latin in the twelfth century by an English scholar. Before it was lost, this trans lated version of Al-Khwarizmi's text found its way to Italy, Spain, and England. Its name, though various modifications, became Alchwarizmi, Al-Karismi, Algoritmi, Algorismi, which named the new art, Algorithm.[11] Thus, Al-Khwarizmi left his name to the history of mathematics in the form of Algorism, the old word for arithmetic.[12]

I-1. Arabic Numerals

Imagine a hillside thousands of years ago. A man emerges from a cave. His brow is heavy, and his arms are long and

8) George N. Atiyah, *Al-Kindi: The Philosopher of the Arabs* (Karachi, Al-Karami Press, 1966), p. 185.
9) Oystein Ore, *Number Theory and its History* (New York, McGraw-Hill Book Company, 1948), p. 185.
10) Florence A. Yeldham, *The Story of Reckoning in the Middle Ages* (London, George C. Harrap and Company, 1926), p. 64.
11) Bodleian Library, Oxford, England, Marsh MSS, 489, fol. 145r~166r.
12) Charles Singer, *A Short History of Scientific Ideas to 1900* (London, Oxford University Press, 1968), p. 162.

muscular. Around his waist he wears a tattered animal skin, and a herd of wild horses passes below him. Back into the cave he rushes and, with grunts and gestures, excitedly tells his clan that 'many, many' horses are passing. This is the best he can count. He has no way of telling them that 30,40 or 50 horses are in the herd, for at best he knows three numbers one, two, and 'many'. Civilizations will rise and fall, and even his own form will change before man learns to count with the ease and exactness of numbers such as 30, 40 or 50. The development of an easy-to-use, easy-to-learn system of numbers was a milestone, reached only after long struggle. In fact, man has had such a system only for about 1,000 years, and mankind has been on earth for a very long time.

In every civlization of which there is historical record there exists some idea of numbers.[13] In the early and more primitive civilizations, this concept is exhibited in a set of number symbols or words.[14] It is common knowledge that the numerals in which the score is given at a football game are called Arabic numerals and one assumes that these have always been in use. In actual fact Europe adopted them from the Muslims only in the thirteenth century. Fighting their introduction and that of the decimal system that went with them for several hundred years, Europe deprived itself of the advantages of one of the world's greatest contibution to mathematics.

Prior to the Arabian numerals, the West relied upon the clumsy system of Roman numerals, and before that upon the even more clumsy Greek numerals. In the decimal system, the number 1843 can be written in four numerals, whereas in the Roman numerals, eleven figures are needed. The result is MDCCCXLIII. It is obvious that even for the result of the simplest arithmetical problem, Roman numerals called for an enormous expenditure of time and labor. The Arabian numerals, on the other hand, render even complicated mathematical tasks relatively simple.[15] Professor J. Houston Banks of Peabody College has stated:

The Roman system seems to have some advantage over the present numerals when we consider the process of addition. Let us consider the addition of 127 and 58 in Roman numerals:

13) David Eugene Smith, ***Number Story of Long Ago*** (Washington, D.C., The National Council of Teachers of Mathematics, 1962), p. v.

14) Howard Franklin Fehr. *A Study of the Number Concept of Secondary School Mathematics* (Ann Arbor, Michigan, Edwards Brothers, 1945), p. 14.

15) Jane Muir, ***Of Men and Number: The Story of the Great Mathematicians*** (New York, Dodd, Mead and Company, 1961), p. 28.

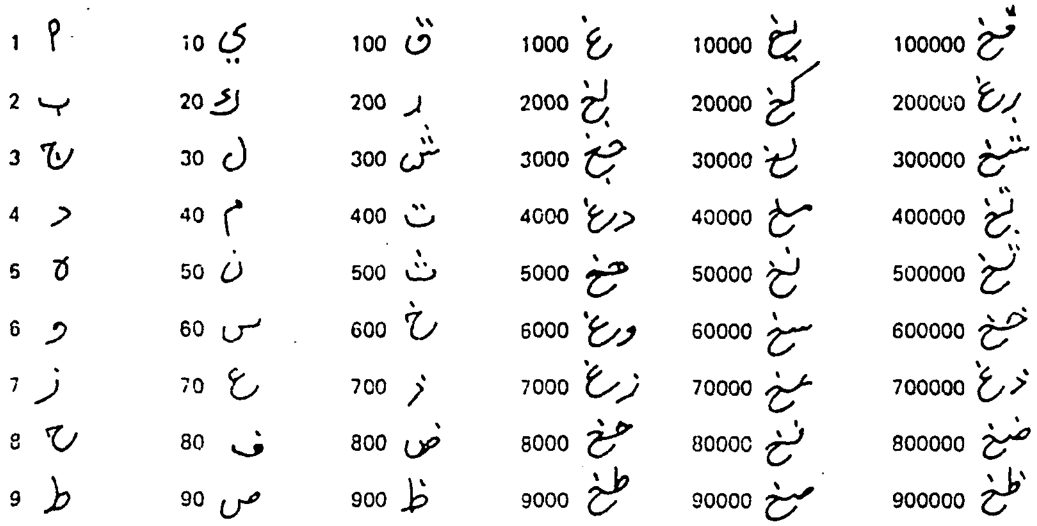

Fig. 3. 1. Early Arabian Numeration using Alphabetic Symbols

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   C X X V I I
     L V I I I
C L X X V V I I I I I
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We can merely bring down all the symbols in each addend; then if we remember that five I's are written as V and two V's as X, the answer becomes CLXXXV. It is not necessary to know the addition combination such as 7+8 and 5+2. However, the process is much longer and more cumbersome. But if we attempt to multiply or divide, it is a different story.[16*]

At the time of the Prophet Mohammed, Arabians had a script which did not differ significantly from that in later centuries. The letters of the early Arabic alphabet were then used as numerals among the Arabians. Arabic alphabetic numerals used before the introduction of the Hindu-Arabic numerals are listed in Figure3. 1.[17]

The numerals 1, 2, 3, 4, 5, 6, 7, 8, and 9 used in almost all parts of the globe are believed by some scholars to be related to nine Sanskrit characters used in ancient times by the Hindus. These numerals were transmitted to Muslims, who modified and introduced them into Europe.[18]

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16) Houston Banks, *Elements of Mathematics*(Boston, Allyn and Bacon, 1969), 3rd edn., pp. 66~67.
* In reading this quotation one has, of course, to be aware that supposition value of Roman numerals is to be considered. The sum of XI and IX is neither XXII which would be 22 nor IIXX which a Roman student would have considered as a funny mispelling of XVIII.

17) Florian Cajori, *A History of Mathematical Notations* (La Salle, Illinois, The Open Court Publishing Company, 1928), Vol. I, p. 29.

18) Mayme I. Logsdon, *A Mathematician Explains*(Chicago, Illinois, The University Press, 1935), p. 43.

The origin of these numeral, which the Muslims themselves call 'Hindi' numerals, is somewhat uncertain and vague. Some writers have suggested that the word 'Hindi' does not necessarily imply that numbers orinated in India, since the Arabians had many meanings for the word. It is perhaps significant that the first Arabic book containing 'Arabic' numerals was written in 874 AD, while the first Indian book containing them appeared two years later.[16]

The Muslim culture brought to the world two great streams of human achievement, a new number language from the East, and the classical mathematics of the West.[20] Nonetheless, whether the numbers were discovered by the Indians or the Muslims, it is indisputably the Muslim mathematicians who made use of them and introduced the decimal system to the world. The ingenious idea of expressing all numbers by means of ten symbols, with each symbol receiving the value of position as well as an absolute value, had escaped both the Alexandrians and the Greeks.

The Arabian system of numeration, based upon the idea of place value, is one of the most rewarding results of human intelligence and deserves the highest admiration. This simplicity of numeration is one of the greatest achievements of the human mind. In the hands of a skilled analyst, it becomes a powerful instrument in 'wresting from nature her hidden truths and occult laws.'[21] Lee Emerson Boyer states:

> Without it, many of the arts would never have been dreamed of, and mathematics would have been still in itscradle. With it, man becomes armed with prophetic power, predicting eclipses, pointing out new planets which the eye of the telescope had not seen, assigning orbits to the erratic wanderers of space, and even estimating the ages that have passed since the universe thrilled with the sublime utterance, 'Let there be light!' Familiarity with it from childhood detracts from our appreciation of its philosophical beauty and its great practical importance. Deprived of it for a short time, and compelled to work with the incon enient method of other systems, we should be able to form a truer idea of the advantages which this

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19) Abdel Salam Said, 'We Remember that Western Arithmetic and Algebra Owe Much to Arabic Mathematicians,' *Arab World*, V, Nos. 1-2, (January 1959), 5.

20) Lancelot Hogben, *Mathematics for the Millions* (New York, W. W. Horton and Company, 1946), p. 235.

21) Lee Emerson Boyer, *Mathematics: A Historical Development* (New York, Henry Holt and Company, 1949), pp. 29~31.

invention has conferred on mankind.[22]

A very powerful system of numeration is used for arithmetic computations. The system evolved slowly and received contributions from many civilizations. It is known, however,as the Arabic system of numeration becuse the Muslims made major contributions. A study of the Arabic system of numeration is worthwhile for the reasons that one will be able to:

1. witness and appreciate the 'beauty' and logic of the system,
2. gain a better understandgin and knowledge of the system used since childhood; and
3. experience the effect that the 'base' has upon this system.[23]

I-2. Muslims Offered the Zero

There is no numeral in the set of Arabic numerals of greater significance than the zero, which is referred to as sifr or 'empty' by the Arabians. While the zero is used as a symbol for nothing, it actually has much meaning. The difference in appearance between 5 and 50 is only a zero, but that small circle is actually one of the world's greatest mathematical innovations. In combination with the nine basic numerals, the zero provides numbers with an infinite variety of values. The zero's creation opened the way for the entire concept of algebraic positive and negative numbers, which are used for calculations, identifcation of electrical charge and discharge, for navigation, etc.

It is interesting to note that while the earliest Hindu example of a zero was found on an inscription of 876 AD at Gwalior, the earliest Muslim zero is contained in a manuscript dated 873 AD. Without the sifr, any number system would be much more complicated and clumsy. It took Europe at least two hundred and fifty years to accept and acknowledge the zero as a gift from the Muslims. The concept of a mathematical representation that appeared to have no content of its own made no sense to European mathematicians. It was not until the late twelfth century that the Europeans really began to make use of the zero and the decimal system.[24]

The Hindus considered a position 'empty' if it were not filled out, and, thus the word sunya (empty or blank) was

22) Ibid.

23) Donald Merrick, *Mathematics for Liberal Arts Students* (Boston, Princle, Weber and Schmidt, 1970), p. 104.

24) Rom Landau, *Arab Contribution to Civilization* (San Francisco, The American Academy of Asian Studies, 1958), p. 29.

25) C. B. Boyer, 'Zero: The Symbol, The Concept, The Number,' *National Mathematics Magazine*, XVIII (1944), pp. 323~30.

used for zero.[25] The Muslims translated the Indian sunya as sifr. When Fibonacci (Leonardo of Pisa) wrote his Liber Abaci in 1202, he spoke of the symbol as Zephirum. A century later Maximus Planudes (1340) called it Tzipha, and this form was still used as late as the sixteenth century. In Italian it was called Zenero, Cenero, and Zephiro. SInce the fourteenth century, zero has been the term used as shown in the records of 1491 by Calnadri and of 1494 by Luca Pacisli. The word nulla appears in Italian translations of Muslim writings of the twelfth century,

	2		3
4		2	
	1		

Fig. 3. 2. Positional Method

and also in the French(Chuquet, Triparty) and German of the fifteenth century. Cipher was still used for zero by Adrian Metiers(1611), Herigone(1634), Cavalieri (1643), and Euler(1783), even though the more modern German word Ziffer had been introduced. The zero symbol is also called cipher and naught, and modern usage permits it to be called 0(word 'O') 'an interesting return to the Greek name omicron.,[26]

Until the intervention of the symbol for zero, it was necessary to have paper or tablets in columns, in order to keep the digits in their proper places. Thus, the numbers shown in Figure 3.2 would stand respectively for 203, 4020, and 100. The purpose of a zero is to keep the other digits in their proper position.[27]

The Hindu symbol for the zero was '⊙' (a dot inside a circle). However, in the Muslim Empire, the Muslim East (Asia, Baghdad and the Muslim West (North Africa, Spain), a number of different representations were used. In the Muslim East, the dot '·' was used and their augmented sets of numerals became: ١, ٢, ٣, ٤, ٥, ٦, ٧, ٨, ٩, ٠. The Muslim West adopted the circle o as their symbol and the complete set of numerals became 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0.[28]

Through the efforts of Muslim scholars, the decimal numerational system was introduced to the civillzed world. It was a system in which the zero was indeed the pivotal point and the elementary operations of arithmetic were tremendously simplified.[29]

I-3. Operation

The Muslims adopted the Greek defin-

26) Marie Haden, 'A History of Our Numerals and Decimal System of Numeration,' (unpublished Master's Thesis, George Peabody College for Teachers, 1931), p. 25.
27) A. Hooper, *The River Mathematics* (New York, Henry Holt and Company, 1945), pp. 13~14.
28) Tawfiq Al-Tawil, *Al-'Arab wa Al-'ilm*(Cairo, Maktabat Al-Nahdah Al-Misriyah, 1968), p. 61.

ition of arithmetical operations but used techniques of their own. Some of the procedures may seem complicated and somewhat unusual because they were based on prior analysis of number construction.[30]

Multiplication

The Hindus' multiplication was very complicated. For instance, for the problem 569 times 5, they generally said:

5.5=25;5.6=30, which changed 25 into 28;

5.9=45. Hence the 0 must be increased by 4. The product is 2845.[31]

However, the Muslims' multiplication was very simple and more readily performed. They used the network or lattice method(shabacah) in which the tablet was divided into squares resembling a chess board. Diagonals were drawn.[32] The multiplication of 239×567, is illustrated in Figure 3.3. To find the product by the use of this device one proceeds as follows. The factors are placed at the top and left of the rectangle. The product for each cell is formed by taking the product of the row and column element for it and recording the digit above and the tens digit below the diagonal. The product of the two original factors is determined by summing the numbers in each diagonal and 'carrying' if necessary.

Division

Fibonacci studied in Muslim schools and in 1202 AD introduced Arabian numerals to Europe.[33] He treated several cases in division, the first being division by a one-digit number. He divided 10004 by 8 as an example by writing the quotient below the divisor and the remainder above the dividend:

4
10004
8
1250

Fibonacci advised one to divide the factors of a number whenever possible, and whenever the divisor is greater than 10, he suggested using the nearest multiple of 10 as a trial divisor. This was taken by him from the Muslims.[34]

The Muslim method of long division,

29) H. E. Slaughter, 'The Evaluation of Numbers—An Historical Drama in Two Acts,' *The Mathematics Teacher*, XXI (October 1928), pp. 307~8.

30) Rene Taton, *History of Science: Ancient and Medieval Science From the Beginnings to 1450* (New York, Basic Books, 1963), Vol. I, p. 406.

31) Florian Cajori, *A History of Mathematics* (New York, The Macmillan Company, 1924), p. 90

32) Robert W. Marks (ed.), *The Growth of Mathematics From Counting to Calculus*(New York, Bantam Books, 1964), p. 100.

33) Philip K. Hitti, *The Near East in History—A 5000-Year Story*(New York, D. Van Nostrand Company, 1960), p. 253.

34) Vera Sanford, *A Short History of Mathematics* (New York, Houghton Mifflin Company, 1930), pp. 100—1.

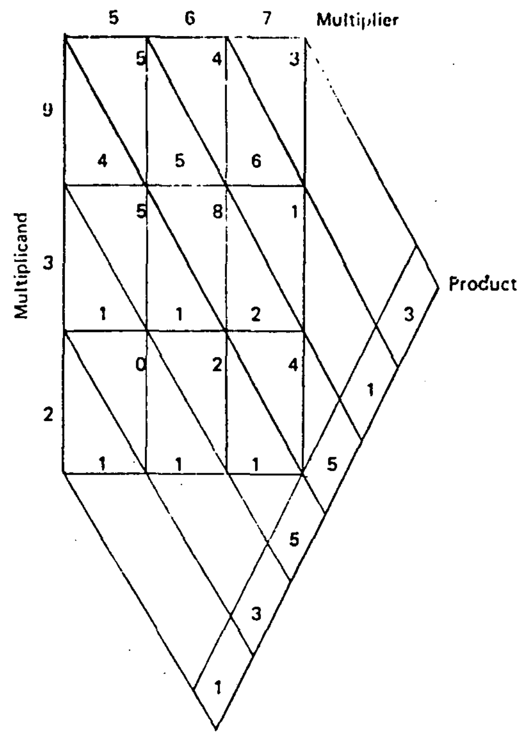
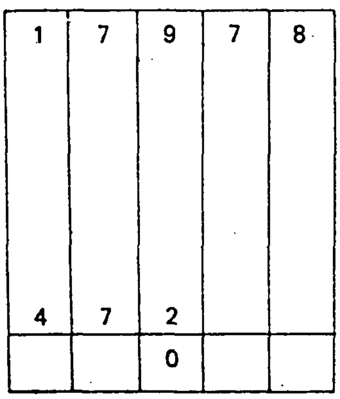
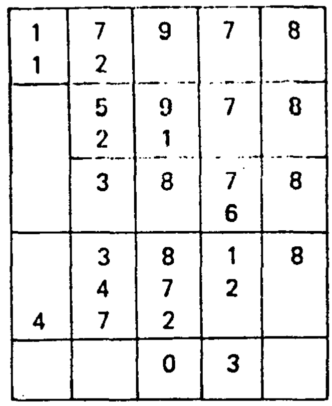


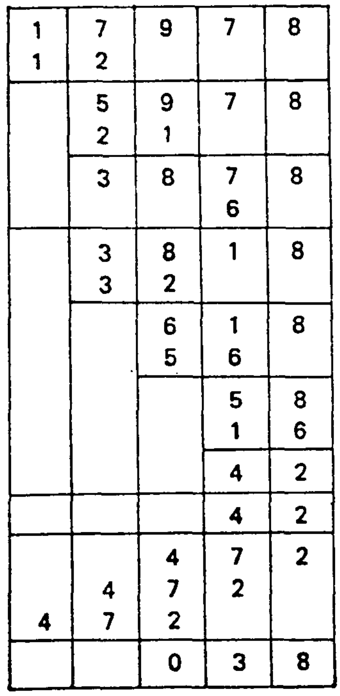
Fig. 3.3. Multiplication by the Lattice Method



A



B



C

Fig. 3.4. An Example of the Arabian Long-Division Method

which required the skill of a mathematical expert, is the oldest method of long division known in the Muslim Empire. An example is presented to illustrate the method. To divide 17978 by 472, a sheet of paper is divided into as many verticl columns as there are digits in the number to be divided. The number to be divided is written at the top of the page and the divisor at the bottom, the first digit of each number being placed at the left-hand side of the paper. Then, taking the left-hand column, 4 divides one zero times; hence, the first digit in the dividend is 0, which is written under the last digit of the divisor. This is represented in Figure 3.4-A. In Figure 3.4-B, the divisor 472 is rewritten immediately above its former position, but shifted one place to the right and the numerals are cancelled. Then 4 evenly divides 17 four times; but, as on trial, it is found that 4 is too large for the first digit of the dividend and 3 is selected. Therefore, 3 is written below the last digit of the divisor and next to the digit of the dividend last found. The process of multiplying the divisor by 3 and subtracting from the number to be divided is indicated in Figure 3.4 and shows that the remainder is 3818. A similar process is then repeated, that is, 373 is divided into 3818, showing that the quotient is 38 and the remainder, 42. This is represented in Figure 3.4-C, which shows the completed process.

I-4. Fractions(Al-Kasr)

The earliest treatment of the subject of common fractions is the Lilavati (1150 AD) of the Hindu mathematician Bhaskara II. Common fractions in Lilavati are denoted by writing the numerator above and the denominator below without any line between them. For example, 3/11 was written as:

3

11

Mixed numbers were written with the integer part above the fraction: thus, $8^3/_4$ was written:

8

3

4

The introduction of the line of separation is due to the Muslims. To denote a fraction in the Muslim method, one writes three-fourths as 3/4; and to denote 3+3/4 one writes the mixed number '$3^3/_4$'.

It is to Muslim mathematicians that credit is due for the first use of decimal fractions. Louis Charles Karpinski wrote:

Our notation of fractions is quite certainly based upon Arabic forms··· The Arabic word for fraction,al-kasr, is derived from the stem of the verb meaning 'to break.' The early writers on algorism commonly used fractio, while Leonardo of Pisa and John of Meurs (fourteenth century) used both fractio and minutum ruptus.[35]

Amicable Numbers

35) Karpinski, op. cit.

A pair of numbers is said to be amicable when the sum of the factors of one is equal to the other, and conversely. Therefore, M and N are amicable numbers when $\sigma_0(N)=M$, and also $\sigma_0(M)=N$.[36] For example, the numbers 220 and 284 are a pair of amicable numbers. The factors of 284 are 1, 2, 4, 71, 142, and their sum is 220. The factors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110, and their sum is 284.

In Muslim mathematical writings the amicable numbers occur repeatedly. They play a role in magic and astrology, in the casting of horoscopes, in sorcery, in talismans.[37] Amicable numbers were one of the hobbies of Abu Zaid Abdel Rahman ibn Khaldun, who was born in Tunis in 1332.[38] Ibn Khaldun wrote that persons who have concerned themselves with talismans affirm that the amicable numbers 220 and 284 have an influence to establish a union or close friendship between two individuals.[39]

The ninth century was a glorious one in Muslim mathematics. for it produced not only Al-Khwarizmi in the first half of the century, but also Thabit ibn Qurra (826-901) in the second half. If Al-Khwarizmi resembled Euclid as 'elementator', then Thabit is the Arabic equivalent of Pappus, the commentator on higher mathematics.[40]

The famous Thabit ibn Qurra also worked on amicable numbers. He was the first writer who reached fame and recognition in this period, and is particularly noted for his translations from Greek to Arabic of works by Euclid, Archimedes, Apollonius, Ptolemy, and Eutocious.[41] Had it not been for his efforts, the number of Greek mathematical works known today would be smaller, as for example, there would have been preserved only the first four rather than the first seven books of Apollonius' Conics. Thabit had so thoroughly mastered the content of the classics he translated that he suggested modifications and generalizations, and a remarkable formula for amicable numbers is credited to him.[42] The formula is as follows:

If p, q, and r are prime numbers, and if they are of the form,

36) Oystein Ore, *Number Theory and Its History* (New York, McGraw-Hill Book Company, 1948), pp. 98—9.

37) Ibid.

38) Muhsin Mahdi, *The Khaldun's Philosophy of History: A Study in the Philosophic Foundation of the Science of Culture* (Chicago, The University of Chicago Press, 1954), p. 27.

39) Leonard Eugene Dickson, *History of the Theory of Numbers: Divisibility and Primality* (Washington, D.C., Press of Gibson Brothers, 1919), Vol. I, p. 36.

40) Carl B. Boyer, *A History of Mathematics* (New York, John Wiley and Sons, 1968), p. 258.

41) George Sarton, *A History of Science* (Cambridge, Mass., Harvard University Press, 1952), Vol. I, p. 446.

42) Boyer, op. cit.

$p=3.2^n-1, q=3.2^{n-1}-, r=9.2^{2n-1}-1,$
then $p, q,$ and r are distinct primes and $2^n pq$ and $2^n r$ are a pair of amicable numbers.[43]

For instance, for $n=2$,

since $p=3.2^n-1$

therefore $p=3.2^2-1=3.4-1=11$

since $q=3.2^{n-1}-1$

$=3.2^{2-1}-1=5$

since $r=9.2^{2n-1}-1$

$=9.2^{4-1}-1=9(8)-1=71$

and since the pair of amicable numbers are $2^n pq, 2^n r$, it follows that, $2^2(11)(5)=220$, and $2^2(71)=284$.

Therefore, it is shown that 220 and 284 are amicable numbers.

Sum of Natural Numbers

Muslim mathematician, Al-Karkhi gave expressions for the sum of the first, second, and third powers of the first n natural numbers as follows:[44]

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

$$1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$$

I-5. Summary

The author believes that arithmetic is a vital part of daily living because it meets practical needs. R. L. Goodstein suggests that without numbers mathematical communication would be exceed ingly cumbersome and compli cated; but, a language lacking expressions for numbers can nonetheless express everything that can be said in the current medium of expression.[45] Conant feels that the Muslims disseminated their own numerical system in such a manner that their influence is detected without difficulty.[46] Outstanding among the Muslim contributions to arithmetic was the development of and introduction into Europe of the Hindu-Arabic numeration system. The contribution of the place holder, zero, the introduction of the modern notation for common fractions, and the use of decimal fractions are only a few of the many contributions made to arithmetic by the Muslims.

II. ALGEBRA

It would be an injustice to pioneers in mathematics to stress modern mathematical ideas with little reference to those who initiated the first and possibly the most difficult steps. Nearly everything

43) Howard Eves, *An Introduction to the History of Mathematics*(New York, Holt, Rinehart dan Winston, 1969), p. 220.

44) W. W. Rouse Ball, *A Short Account of the History of Mathematics* (New York, Dover Publications, 1960), pp. 159~60.

45) R. L. Goodstein, 'The Arabic Numerals, Numbers, and the Definition of Counting,' *Mathematical Gazette*, XL (May 1965), p. 129.

46) Levi Leonard Conant, *The Number Concept* (New York, Macmillan and Company, 1923), p. 70

useful that was discovered in mathematics before the seventeenth century has either been so greatly simplified that it is now part of every regular school course, or it has long since been absorbed as a detail in some work of greater generality.[1]

The Muslims translated numerous Greek works in mathematics as they did in other fields of science. At the same time, they turned to the East and gathered all that was available in India in the way of science, and particularly in mathematics.[2] In the field of algebra the Muslims soon made original contributions which proved to be the greatest of their distinctive achievements in mathematics.[3]

II-1. Definition of Algebra

Algebra is that 'branch of mathematical analysis which reasons about quantities using letters to symbolize them.'[4] Algebra is defined in the *Mathematics Dictionary* as 'a generalization of arithmetic; e.g., the arithmetic facts that $2+2+2=3\times2, 4+4+4=3\times4$, etc., are all special cases of the (general) algebraic statement that $x+x+x=3x$, where x is any number.'[5] The Muslim scholar ibn Khaldun defined algebra as a 'subdivision' of arithmetic. This is a craft in which it is possible to discover the unknown from the known data if there exists a relationship between them.[6]

In the ninth century the Muslim mathematician Al-Khwarizmi wrote his classical work on algebra, *Al-Jabr wa-al-Mugabala*. In this title the word *Al-Jabr* means transposing a quantity from one side of an equation to another, and *Mugabala* signified the simplification of the resulting expressions.[7] Figuratively, *al-jabr* means restoring the balance of an equation by transposing terms.[8] Because of the double title, the explanation given contains a comment upon the second word, *al-maqabala*, as well as the first one. According to David Eugene Smith:

In the 16th century it is found in Eng-

1) E.T. Bell, *Men of Mathematics* (New York, Simon and Schuster, 1937), p. 14.
2) Daoud S. Kasir, *The Algebra of Omar Khayyam*(New York, J. J. Little and Ives Company, 1931), p. 16.
3) Bodleian Library, Oxford, England, Huntingdon MSS, 214, fol. ff34r-52r.
4) Isaac Funk, Calvin Thomas, and Frank H. Vizetelly(Supervisors), *New Standard Dictionary of the English Language* (New York, Funk and Wagnallis Company, 1940), p. 70.
5) Glenn James and Robert C. James (eds.), *Mathematics Dictionary* (New York, D. Van Nostrand Company, 1963), p. 17.
6) Franz Rosenthal, trans., *The Migaddimah Ibn Khaldun: Autobiography*(New York Bollingen Foundation, 1958), Vol. III, p. 126.
7) Isma'il Mazhar, *Tarikh al-Fikr al-'Arabi: fi Nushu' ih wa Tatwirih bi Ttarjamah wa Annagil 'an al-Hadar ah Al-Yunaniyah*(Cairo, Dar al-'Uswarli Itab' wa Annashur bi Masr).
8) Morris Kline, *Mathematics and the Physical World*(New York, Thomas Y.Crowell Company, 1959), p. 69,

lish as *algiebar* and *almachabel*, and in various other forms but was finally shortened to *algebra*. The words mean restoration and opposition, and one of the clearest explanations of their use is given by Beha Eddin(1600 AD) in his *Kholasat Al-Hibas* (essence of arithmetic): The member which is affected by a minus sign will be increased and the same added to the other member, this being algebra; the homogeneous and equal terms will then be cancelled, this being *Al-Muqabala*.[9]

That is, given $x^2+5x+4=4-2x+5x^3$,

Al-Jabr gives $x^2+7x+4=4+5x^3$,

Al-Muqabala gives $x^2+7x=5x^3$.

Therefore, the best translation for *Hisab Al-Jabr Wa-al-Muqabala* is 'the science of equations.'[10]

II-2. The Origin of the Term 'Algebra'

Al-Khwarizmi's text of algebra, entitled *Al-Jabr Wa-Al-Muqabala* (the science of cancellation and reduction) was written in 820 AD.[11] A Latin translation of this text became known in Europe under the title *Al-Jabr*.[12] Thus 'the Arabic word for reduction, *al-Jabr*, became the word *algebra*.'[13]

II-3. Al-Khwarizmi

From the eighth to the thirteenth centuries, the center of scientific activity was Arabia. Scientific activity was centered in the Muslim world, especially at the court of the Caliph Al-Ma'mum.[14] It was there that Al-Khwarizmi (825 AD) influenced mathematical thought more than any other medieval writer by finding a system of analysis for solving equations of first-and second-degree in one unknown by both algebraic and geometric means.[15]

The first half of the ninth century is characterized by Sarton in his *Introduction to the History of Science* as 'the time of Al-Khwarizmi,' because he was 'the gre-

9) David Eugene Smith, *History of Mathematics* (New York, Ginn and Company, 1925), Vol. II, p. 388.
10) John K. Baumgart, 'History of Algebra,' *Historical Topics for the Mathematics Classroom*, Thirty-First Yearbook of the National Council of Teachers of Mathematics (Washington, D.C., National Council of Teachers of Mathematics, 1969, pp. 233-4.
11) Sidney G. Hocker, Wilfred E. Barnes, and Calvin T. Long, *Fundamental Concepts of Arithmetic* (Englewood Cliffs, N.J., Prentice-Hall, 1963), p. 9.
12) H.A. Freebury, *A History of Mathematics: For Secondary Schools* (London, Cassel and Company, 1968), p. 77.
13) Solomon Gandz, 'The Origin of the Term Algebra,' *The American Mathematical Monthly*, XXXIII (May 1926), 437.
14) Edna E. Kramer, *The Nature and Growth of Modern Mathematics* (New York, Hawthorn Books, 1970), p. 85.
15) Franklin W. Kokomoor, *Mathematics in Human Affairs* (New York, Prentice-Hall, 1946), p. 172.

atest mathematician of the time, and if one takes all circumstances into account, one of the greatest of all times.'[16] E. Wiedmann has said: 'His works, which are in part important and original, reveal in Al-Khwarizmi a personality of strong scientific genius.'[17]

David Eugene Smith and Louis Charles Karpinski characterized Al-Khwarizmi as:

...the great master of the golden age of Baghdad, one of the first of the Muslim writers to collect the mathematical classics of both the East and the West, preserving them and finally passing them on to the awakening Europe. This man was... a man of great learning and one to whom the world is much indebted for its present knowledeg of algebra and of arithmetic.[18]

It was Mohammad Khan who stated:

In the foremost rank of mathematicians of all times stands Al-Khwarizmi. He composed the oldest works on arithmetic and algebra. They were the principal source of mathematical knowledge for centuries to come both in the East and the West. The work on arithmetic first introduced the Hindu numbers to Europe, as the very name algorism signifies; and the work on algebra not only gave the name to this important branch of mathematics in the European world, but contained in addition to the usual analytical solution of linear and quadratic equations (without, of course, the conception of imaginary quantities) graphical solution of typical quadratic equations.[19]

The mathematics that the Muslims inherited from the Greeks made the division of an estate among the children extremely complicated, if not impossible. It was the search for a more accurate, comprehensive, and flexible method that led Al-Khwarizmi to the innovation of algebra.[20] While engaged in astronomical work at Baghdad and Constantinople, he found time to write the algebra which brought him fame.[21] His book, Al-Kitab al-Mukhtasar fi hisab al-jabr wa-al-Muqabala, is devoted to finding solutions to

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16) George Sarton, *Introduction to the History of Science* (Baltimore, The Williams and Wilkins Company, 1927), Vol. I, p. 563.

17) M. Th. Houtsma, T,W. Arnold, R. Basset, and R. Hartmann (eds.), *The Encyclopaedia of Islam* (London, Luzac and Company, 1913), Vol. I, p. 912.

18) David Eugene Smith and Louis Charles Karpinski, *The Hindu-Arabic Numbers* (Boston and London, Ginn and Company, 1911), pp. 4-5.

19) Mohammad Abdur Rahman Khan, *A Brief Survey of Moslem Contribution to Science and Culture* (Lahore, Sh. Umar Daraz at the Imperial Printing Works, 1946), pp. 11-12.

20) Rom Landau, *The Arab Heritage of Western Civilization* (New York, Arab Information Center, 1962), pp. 33-4.

21) Florence Annie Yeldham, *The Story of Reckoning in the Middle Ages* (London, George G. Harrap and Company, 1926), p. 64.

practical problems which the Muslims encountered in daily life.[22]

In evolving his algebra, Al-Khwarizmi transformed the number from its earlier arithmetical character as a finite magnitude into an element of relation and of infinite possibilities. It can be said that the step from arithmetic to algebra is in essence a step from 'being' to 'becoming' or from the static universe of the Greek to the dy namic ever-living, God-permeated one of the Muslims.[23]

Al-Khwarizmi emphasized that he wrote his algebra book to serve the practical needs of the people concerning matters of inheritance, legacies, partition, lawsuits, and commerce. He dealt with the topic which in Arabic is known as 'ilm al-fara'id[24] (the science of the legal shares of the natural heirs).[25] Gandz stated in The Source of Al-Khwarizmi's Algebra:

Al-Khwarizmi's algebra is regarded as the foundation and cornerstone of the sciences. In a sense, Al-khwarizmi is more entitled to be called 'the father of algebra' than Diophantus because Al-Khwarizmi is the first to teach algebra in an elementary form and for its own sake, Diophantus is primarily concerned with the theory of numbers.[26]

In the twelfth century the algebra of Al-Khwarizmi was translated into Latin by Gerhard of Cremona and Robert of Chester.[27] It was used by Western scholars until the sixteenth century.[28] Of the translation by Robert of Chester, Sarton judicially remarked: 'The importance of this particular translation can hardly be exaggerated. It may be said to mark the beginning of European algebra.'[29]

After Al-Khwarizmi, there were many other Muslims who studied and taught algebra, but they made few new discov-

22) Lancelot Hogben, *Mathematics for the Million*(New York, W.W. Norton and Company, 1946), pp. 290-1.
23) Rom Landau, *Arab Contribution to Civilization* (San Francisco, The American Acadamy of Asian Studies, 1958), p. 33.
24) Bodleian Library, Oxford, England, Marsh MSS, 640, fol. (f. 102).
25) Solomon Gandz, 'The Algebra of Inheritance,' *Osiris*, V(1938), 324.
26) Solomon Gandz, 'The Source of Al-Khwarizmi's Algebra', *Osiris*(Bruges, Belgium, The Saint Catherine Press Ltd., 1936), Vol. I, p. 264.
27) Ibid., p. 263.
28) Joseph Hell, *The Arab Civilization*(Lahore, Sh. Mohd. Ahmad at the Northern Army Press, 1943), p. 95.
29) George Sarton, *Introduction to the History of Science* (Baltimore, The Williams and Silkins Company, 1953), Vol. II, Part I, p. 176.
30) Walter H. Carnahan, 'History of Algebra,' *School Science and Mathematics*, XLVI, 399 (January 1946), 10.

eries. they were content to know what he had written in his great book.[30]

II-4. Roots

The term 'root' had its origin in the Arab language. 'Latin works translated from the Arabic have radix for a common term, while those inherited from the Roman civilization have latus.' Radix (root) is the Arabic jadhr, while latus referred to the side of a geometric square.[31]

The Arabic word for root was used by Al-Khwarizmi to denote the first-degree term of a quadratic equation. Ai-Khwarizmi wrote, 'the following is an example of squares equal to roots: a square is equal to 5 roots. The root of the square then is 5, and 25 forms its square, which of course equals 5 of its roots.'[32]

Square Root

The method of extracting the square root employed by the Muslims resembled their method of division. For example, to find the square root of 107584 vertical lines are drawn and numerals are partitioned into periods of two digits. See details in Figure 4.1. The nearest root of 10 is 3, which is placed both below and above, and its square, 9, is subtracted from 10. The 3 is now doubled and the result is written in the next column. Six is contained twice in 17, the remainder with first figure of the next period. The 2 is set down both above and below, and being multiplied by 6 gives 12, which subtracted from 17, leaves 5. The square of 2, or 4, is now subtracted from 55. The difference 51, together with the succeeding figure, or 518, is divided by the double of 32, or 64, giving 8 for the quotient. Then 8 times 64, or 512, is subtracted from 518 with a difference of 6. This digit together with the succeeding figure forms 64 which is exhausted by subtracing from it the square of 8. Therefore, the square root of 107584 is 328. It has been said that this method was adopted from the Muslims by the Hindus.[33]

Al-Karkhi, another Muslim mathematician, employed a method of approximation to find the square root of numbers using the formula.[34]

$$\sqrt{a}=w+\frac{a-w^2}{2w+1}$$

31) Solomon Gandz, 'Arabic Numerals,' *American Mathematical Monthly*, XXXIII(January 1926), 261.
32) Philip S. Jones, '"Large" Roman Numerals,' *The Mathematics Teacher*, CXVII (March 1954), 196.
33) Indian Office Library, London, England, Arabic MSS, 757, fol. 4^b-5^a.
34) George E. Reves, 'Outline of the History of Algebra" *School Science and Mathematics*, III (January 1952), 63.

1	0	7	5	8	4
	9				
	1	7			
	1	2			
		5	5		
			4		
		5	1	8	
		5	1	2	
				6	4
				6	4
				0	0
		6	6		
	3		2		

Fig. 4. 1. An Illustration of the Arabian Method for Extracting Square Root

The approximate root for $\sqrt{17}=4=w$. Therefore

$$\sqrt{17}=4+\frac{17-16}{2(4)+1}=4\frac{1}{9}=4.1111$$

and this value checks fairly well with 4.123106, the value precise to six figures.

II-5. Linear and Quadratic Equations

The Egyptians solved equations of the first degree more than four thousand years ago. That is, they found that the solution of the equation $ax+b=0$ is $x=-b/a$. The graph of the equation is represented in geometry by a straight line. The quadratic equation, however, $ax^2+bx+c=0$ was solved by the Muslims with the formula:

$$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$$

The various conic sections, such as the ellipse, the parabola, and the hyperbola are the geometric representations of quadratic equations in two variables which were studied by the Muslims.[35]

In his work on linear and quadratic equations. Al-Khwarizmi used special technical terms for the various multiples or powers of the unknown. The unknown is referred to as a 'root' and the unknown squared is called the 'power.' With this vocabulary, Al-Khwarizmi would describe general linear equations as 'roots equal to numbers.' In present-day notation it would appear as $bx=c$. Instances of linear equations are: one root equals three, $x=3$; four roots equal twenty, $4x=20$; and one-half a root equals ten $(1/2)x=10$.[36]

Al-Khwarizmi separated general quadr-

35) Edward Kasner and James Newman, *Mathematics and the Imagination* (New York, American Book Stanford Press, 1945), p. 17.

36) Ali Mustafa Mashrafah Wa Mohammed Musa Ahmad (ed.), *Kitab Al-Jabr wa-Al-Muqabala Li Mohammed Ibn Musa Al-Khwarizmi* Cairo, Dae Al-Katib Al-Arabi Littibah wa Al-Nashr, 1968), p. 16.

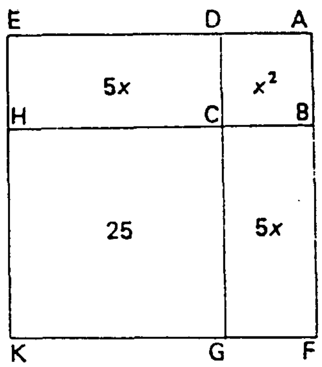

Fig. 4.2. Paradigm of Al-Khwarizmi's for Quadratic Equation $x^2+10x=39$

atic equations into five cases for purposes of finding the solution to a given equation. The five cases he considered were: (1) squares equal to roots, $ax^2=bx$; (2) squares equal to numbers, $ax^2=b$; (3) squares and roots equal to numbers, $ax^2+bx=c$; (4) squares and numbers equal to roots, $ax^2+c=bx$; and (5) squares equal to roots and numbers, $ax^2=bx+c$. In all applications, Al-Khwarizmi considered a,b,c positive integers with a=1. He was concerned with only positive real roots, but he recognized the existence of a second root which was not conceived of previously.[37] Examples are given of cases (3), (4), and (5) above to illustrate the methods of Al-Khwarizmi.

Case (3) : Square and roots equal to numbers, $x^2+10x=39$.

Construct the square ABCD with side $AB=x$. Extend AD to E and AB to F such that DE=BF=(1/2)10=5, then complete the square AFKE. By extending DC to G and BC to H, the area of square AFKE may be expressed as $x^2+10x+25$. However, the equation to be solved is $x^2+10x=39$. Therefore, 25 must be added to each member of this equation to yield $x^2+10x+25=64$, which is the required area. In other words, $x^2+10x+25$ is a perfect square $(x+5)^2$, and this is equal to another perfect square, 64. Hence, the dimensions of the area $(x+5)^2$ must by 8 by 8. However, since $AF=x+5=8$. this means $x=3$.[38]

Case (4) : Squares and numbers equal to roots, $x^2+21=10x$, $x<b/2$, where a is coefficient of x

Construct a rectangle ABCD with side $AB=x$ and BC=10. The area of rectangle $ABCD=10x=x^2+21$. On side BC mark off a point E such that BE=BA, then complete the square ABEF. It follows that the area of rectangle CDFE=21. Let H be the midpoint of BC. Extend side CD to N such that CN=CH=5 and com plete the square HCNM, whose area is 25. From I, the midpoint of AD, construct the point S such thet $IS=IF=5-x$ and

37) Aydin Sayili, *'Abd al-Hamid ibn Turk and the Algebra of His Time* (Ankara, Turk Tarih Kuruma Basimeni, 1962), p. 146.

38) Walter H. Carnahan, 'Geometric Solutions of Quadratic Equations,' *School Science and Mathematics*, XLVII, No. 415 (November 1947), pp. 689-90.

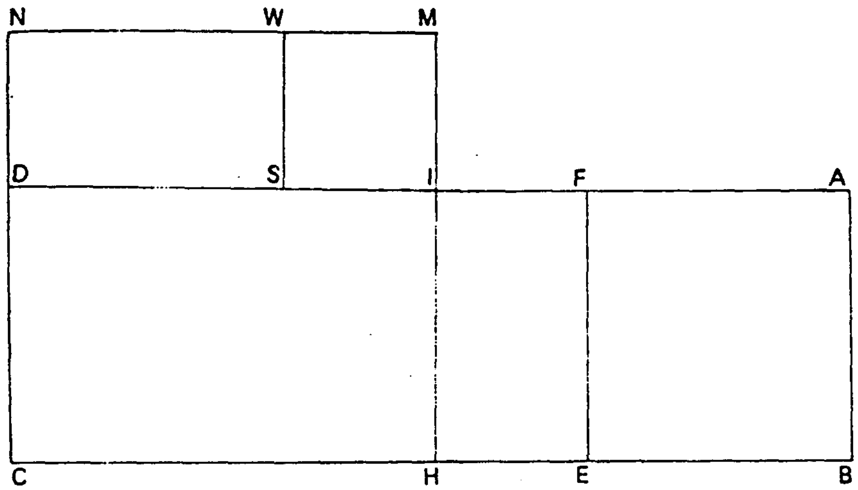

Fig. 4. 3. Paradigm of Al-Khwarizmi for Quadratic Equation $x^2+21=10x$

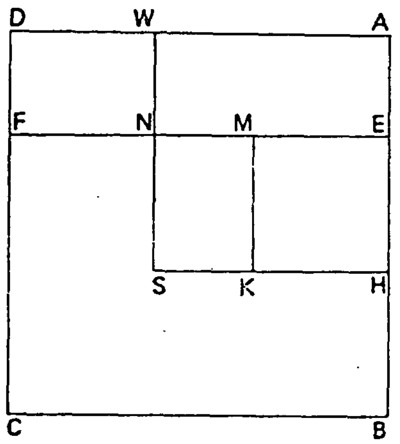

Fig. 4. 4. Paradigm of Al-Khwarizmi for Quadratic Equation $x^2=3x+4$

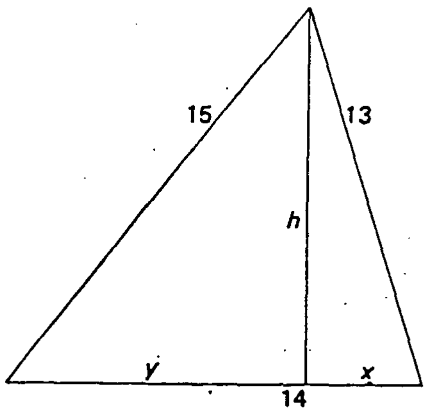

Fig. 4. 5. Paradigm for Al-Khwarizmi's Triangle

complete the square MISW, whose area is $(5-x)^2$. Since DS$=x$, the area of rectangle DSWN$=x(5-x)$=the area of rectangle FEHI. Therefore, the area of rectangle CDIH plus the area of rectangle DSWN $=21$. Thus the area of square HCNM= the area of rectangle CDFE+ the area of square WSIM$=21+(5-x)^2=$ 4 or $x=3$.[39]

Case (5) : Square equal to roots and numbers, $x^2=3x+4$

Construct square ABCD with sides$=x$. Select a point E on side AB such that

39) Martin Levey, *The Algebra of Abu Kamil* (Madison, The University of Wisconsin Press, 1966), pp. 23-4.

BE=3 and complete BEFC. The area of rectangle BEFC=$3x$ and the area of rectangle AEFD=4. Let H bisect segment EB, and construct square EHKM with area=9/4. By extending HK to S such that KS=AE=DF and constructing SW perpendicular to DA, rectangles MKSN and DWNF have equal areas. The equality of these areas follows from DW =HB=HE=KM. The area of square AHSW is (SENW+MKSN)+EHKM or 4 +9/4. The side AH=5/2 and the side A B=AH+HB or 5/2+3/2, therefore $x=4$.[40]

Al-Khwarizmi gave an algebraic method for finding the altitude and the two segments of the base formed by the foot of the altitude, x and y, of the triangle when the three sides (13, 14, and 15) are given, as in Figure 4.5, The square of the height, h^2, is equal to $13-x^2=15^2-y^2=15-(14-x)^2$. Hence $169-x^2=225-196+28x-x^2$. And upon simplification, $x=5$; and it follows that $h^2=169-25$ and that $h=12$.[41]

The change from the Greek conception of a static universe to a new dynamic one was initiated by Al-Khwarizmi who was the herald of modern algebra, and the first mathematician to make algebra an exact science. After dealing with equations of second degree, Al-Khwarizmi discussed algebraic multiplication and division.[42]

II-6. Miscellaneous

Following the period of Al-Khwarizmi's works came those of Thabit ibn Qurra (836~901 AD), mathematician and linguist. His chief contribution to mathematics was in his translations of Euclid, Archimedes, Apollonius, and Ptolemy. Fragments of some original writing in the area of algebraic geometry have also been preserved. This particular branch of algebra received considerable attention from Muslim mathematicians.[43] According to Karl Fink:

Al-Khwarizmi calls a known quantity a number, the unknown quantity jidr (root) and its square mal (power). In Al-Karkhi we find the expression kab(cube) for the third power, and there are formed from these expressions mal mal=x^4, mal kab =x^5, kab kab=x^6, mal mal kab=x^7, etc.[44]

40) Louis Charles Karpinski, trans, *Robert of Chester's Latin Translation of Algebra of Al-Khwarizmi* (London, Macmillan and Company, 1915), p. 87.
41) Henrietta O. Midonick (ed.), *The Treasure of Mathematics*(New York, Philosophical Library, 1965), pp. 432-3.
42) Landau, op. cit., pp. 31-2.
43) Lynn Thorndyke, *A Short History of Civilization* (New York, F.S. Crofts and Company 1930), p. 292.

In his History of Mathematics, David Eugene Smith stated:

...Al-Haitham of Basra, who wrote on algebra, astronomy, geometry, gnomic, and optics attempted the solution of the cubic equation by the aid of conics...[45]

The Muslims discovered the theorem that for integers the sum of two cubes can never be a cube. This theorem was later rediscovered by P. Fermat, a French physician, and claimed by him. Creditable work in number theory and algebra was done by Al-Karkhi of Baghdad, who lived at the beginning of the eleventh century. His treatise on algebra is sometimes considered the greatest algebraic work of Muslim mathematicians; it shows the influence of Diophantus. For the solution of quadratic equations, he gave both arithmetical and geometrical proofs.[46]

Al-Karkhi's work contains basic algebraic theory with application to equations and especially to problems to be solved for positive rational numbers. For instance, to find two numbers the sum of whose cubes is a square number yields the algebraic expression: $x^3+y^3=z^2$. To solve the equation in rational numbers, let:

$$y=mx, z=nx; x^3+m^3x^3=n^2x^2, x^3(1+m^3)=n^2x^2$$

By cancellation of x^2, therefore,

$$x=\frac{n^2}{1+m^3}$$

where m and n are arbitrary positive rational numbers.[47] As a special solution, Al-Karkhi gave the following values: $x=1$, $y=2$, $z=3$. The same method is clearly applicable to many more general rational problems having the form $ax^n+by^n=cz^{n-1}$.[48]

One of the oldest methods for approximating the real root of an equation $ax+b=0$ is often called the rule of double false position. The Muslims called the rule hisab al-Khataayn; it is found in the works of Al-Khwarizmi. This rule seems to have come from India, but it was the Muslims who made it known to European scholars. In order to explain the rule, let g_1 and g_2 be any guessing values of x, and let f_1 and f_2 be the errors. Therefore, if the guesses were right, then ag_1+

44) Karl Fink, *A Brief History of Mathematics* (The Open Court Publishing Company, 1900), p. 75.

45) David Eugene Smith, *History of Mathematics* (New York, Dover Publications, 1958), Vol. I, pp. 175–6.

46) Florian Cajori, *A History of Mathematics* (New York, The Macmillan Company, 1924), p. 106.

47) Howard Eves, *An Introduction to the History of Mathematics* (New York, Holt, Rinehart and Winston, 1969), 3rd edn., p. 201.

48) Oystein Ore *Number Theory and Its History*(New York, McGraw-Hill Book Company, 1948) pp. 185–7.

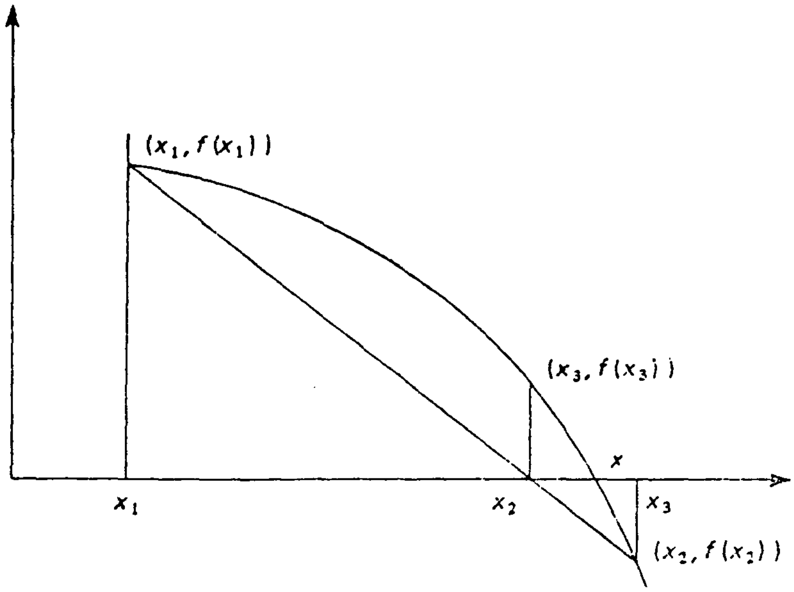

Fig. 4.6. Rule of Double False Position of Al-Khwarizmi

$b=0$; $ag_2+b=0$. However, if the guesses were wrong, then

$ag_1+b=f_1$ (1)

$ag_2+b=f_2$ (2)

$a(g_1-g_2)=f_1-f_2$ (3) by subtraction of (2)from(1)

From (1)

$ag_1g_2+bg_2=f_1g_2$

and from(2) by subtraction

$ag_1g_2+bg_1=f_2g_1$

Therefore

$b(g_2-g_1)=f_1g_2-f_2g_1$ (4)

Dividing (4) by (3)

$$\frac{b(g_2-g_1)}{a(g_1-g_2)}=\frac{f_1g_2-f_2g_1}{f_1-f_2}$$

or

$$\frac{-b}{a}=\frac{f_1g_2-f_2g_1}{f_1-f_2}$$

But, since

$$\frac{-b}{a}=x\ ,$$

therefore,

$$x=\frac{f_1g_2-f_2g_1}{f_1-f_2}$$

Suppose, for example, that $2x-5=0$; guessing value for $x: g_1=5,\ g_2=1$.

Then, $2.5-5=5=f_1$ and $2.1-5=-3=f_2$

But

$$x=\frac{f_1g_2-f_2g_1}{f_1-f_2}=\frac{5.1-(-3)5}{5-(-5)}=\frac{20}{8}$$
$=2.5$[49]

According to Howard Eves, this method was used by the Muslims and can be illustrated geometrically by letting x_1 and x_2 be two numbers lying close to and on each side of a solution x of the equation $f(x)=0$. The intersection with the x-axis

49) Smith, *History of Mathematics*, op. cit., Vol. II, pp. 437-9.

of the chord joining the point$(x_1, f(x_1))$, $(x_1, f(x_2))$ gives a better approximation to the solution:

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

The process can now be applied with appropriate pairs x_1, x_3 or x_3, x_2 depending on circumstances.[50] This is the numerical method of 'False Positions'(Regular Falsi) which is used in numerical analysis today.

II-7. Summary

The Muslims not only created algebra, which was to become the indispensable instrument of scientific analysis, but they laid the foundations for methods in modern experimental research by the use of mathematical models. Since Mohammed ibn Musa Abu Djefar Al-Khwarizmi was the founder of the Muslim school of mathematics, the subsequent Muslim and early medieval works on algebra were largely founded on his algebraic treatise. Al-Khwarizmi's work plays an important role in the history of mathematics, for it is one of the main sources through which Arabic numerals and Muslim algebra came to Europe.

The contribution of Muslim mathematicians to the field of algebra includes methods for finding the solution to linear and quadratic equations. Solutions to these equations were also given by geometric methods. Al-Karkhi contributed rational solutions to certain special equations of degree higher than two and a method for approximating the solution to linear equations. These are but a few of the more outstanding developments in algebra that resulted directly from the efforts of Muslim mathematicians.

III. TRIGONOMETRY

The Muslim mathematicians' great interest in arithmetic, number theory, and algebra appears to have led them into related areas of the applied and theoretical sciences. They had more than passing interest in the works of earlier civilizations. One can see something of this interest in the fact that they translated virtually all known information of their day into Arabic. The scholarly endeavors of search, translation, and research were in evidence in many of the more important centers of Muslim learning.

Arithmetic, and its application to the commercial and business needs of Muslim life, became the impetus for further study and research into mathematics. It seems natural that the Muslims should turn early to an investigation of the fields of astrology and astronomy. Here, in addition to arithmetic, they needed the

50) Eves, op.cit., p. 203.

use of trigonometry to present a clear model of the heavens and its relationships to their mode of life.

Trigonometry, the handmaiden of astronomy, became an absorbing study for Muslim mathematicians, and it led to several useful and well-known studies in the sciences. For example, the Muslims appear to have been the first to give serious study to the principles of light. Al-Haitham wrote an important treatise on optics which became a classic for several centuries. In his treatise, Al-Haitham set forth the early form of what was to become Snell's law of refraction of light. Al-Haitham's Optics inspired attention to astronomy and trigonometry and formed the basis of scholarly research during the Dark Ages and Middle Ages. There is little doubt that these investigations provided considerable support for such men as Leonardo da Vinci, Galieo, and Newton.

The initial development of trigonometry was evidently motivated by a need for numerical solutions to problems related to spherical astronomy. Its emergence as a branch of mathematics, independent of astronomy, was surely a slow process.[1] Perhaps more than any other branch of mathematics, trigonometry appears to have developed as a result of a continual interplay between the supply of applicable mathematical theories and the demands of the science of astronomy.[2] Their relationship was considered intimate until the Renaissance (1450~1600 AD), during which time trigonometry was treated as an auxiliary topic to astronomy, while the problems of mathematical astronomy were related to spherical trigonometry.[3]

After the decline of Alexandria, Greek science lingered only in southern Italy and in Byzantium but was later revived and spread by the Muslims. In the ninth and tenth centuries, while Europe was still in relative darkness, Muslim science and culture were at their height.[4] An investigation of the development of trigonometry during the twelfth and thirteenth centuries will serve as an indication of the progressive work in science conducted by the Muslims.[5]

1) Abbas El-Azzawi, *History of Astronomy in Iraq*(Baghdad, Iraq Academy Press, 1959), p. 17
2) Edward S. Kennedy, 'The History of Trigonometry', *Historical Topics for the Mathematics Classroom*, Thirty-first Yearbook National Council of Teachers of Mathematics (Washington, D.C., National Council of Teachers of Mathematics, 1969), p. 333.
3) George Sarton, *The Appreciation of Ancient and Medieval Science During the Renaissance* (1450-1600) (Philadelphia, University of Pennsylvania Press, 1955), p. 160.
4) H.T. Pledge, *Science Since 1450: A Short History of Mathematics, Physics, Chemistry, and Biology* (New York, Philosophical Library, 1947), p. 11.
5) George Sarton, *Introduction to the History of Science: From Rabbi Ben Ezra to Roger*

III-1. Definition

The fundamental idea of trigonometry is the measurement of distances indirectly. It would be a physical impossibility to measure directly the height of the great pyramid in Egypt or other inaccessible distances, such as the width of gorge to be bridged. These and many other problems in the field of surveying and navigation depend upon the solution of triangles.[6]

The word trigonometry comes from the Greek words tri, meaning 'three'; gonon, 'angle'; and metria, 'measurement.' These terms explain the primary signiicfance of this branch of mathematics.[7] George Howe defined trigonometry as 'the science of angles; its province it to teach how to measure and employ angles with the same ease that we handle lengths and areas.'[8] Trigonometry has been defined also as the measurement and calculations of the sides and angles of a triangle.[9]

Although the term trigonometry was not used until 1595, when it was initially introduced by Pitiscus,[10] the Muslims had worked diligently and meticulously on the early development of the science.[11] The following sections will present the Muslim contributions to trigonometry.

III-2. The Origin of Trigonometry

Trigonometry as it is known today as a branch of mathematics that is linked with algebra. As such, it is to be dated back to the eighth century. When treated purely as a development of geometry, however, it is dated to the time of the great Greek mathematicians and astronomers who flourished about two hundred years before and after the beginning of the Christian era. If regarded simply as 'tri-angle-measurement,' which is all the word trigonometry implies, its roots go back to the Egyptian period four thousand years ago.[12]

By the middle of the twelfth century, Latin mathematicians were acquainted with Muslim trigonometry, if not in its very latest state, at least in the one it had reached before the end of the preceding

Bacon (Baltimore, The Williams and Wilkins Company, 1953), Vol. II, Part I, p. 11.

6) Lee Emerson Boyer. *Mathematics: A Historical Development* (New York, Henry Holt and Company, 1954), p. 415.

7) A. Hooper, *The River Mathematics* (New York, Henry Holt and Company, 1945), p. 222.

8) George Howe, *Mathematics for the Practical Man* (New York, D. Van Nostrand Company, 1957), p. 81.

9) Charles Hutton, *A Course of Mathematics* (Glasgow, Richard Griffin and Company, 1833), p. 415.

10) Alfred Hooper, *Makers of Mathematics* (New York, Random House, 1948), p. 107.

11) Josper O. Hassler and Rolland R. Smith, *The Teaching of Secondary Mathematics* (New York, The Macmillan Company, 1935), pp. 87-8.

12) The Faculties of University of Chicago, Editorial Advisors, *Encyclopaedia Britannica*(Chicago, Encyclopaedia Britannica, 1969), Vol. 22, pp. 235-6.

century.[13] Practically all of the advanced trigonometrical work of the twelfth and thirteenth centuries was produced by Muslim mathematicians and written in the Arabic language. Latin trigonometry was a pale reflection of Muslim trigonometry at this time,[14] and it would wait until the fourteenth century before gaining importance at Merton College in Oxford.[15]

The trigonometry of Musilms is based on Ptolemy's theorem but is superior in two important respects. It employs the sine where Ptolemy used the chord and is in algebraic instead of geometric form.[16] The theory of sine, cosine, and tangent is a legacy of the Muslims. The brilliant epochs of Peurbach, Regiomontanus, and Copernicus cannot be recalled without being reminded of the fundamental and preparatory labors of the Muslim mathematicians.[17]

Consider the movement of a line (now known as the radius vector) in a counterclockwise direction round a fixed point. The perpendiculars drawn from the end point of this line, in its various positions, to the original direction form segments which correspond to the half-chords referred to by Ptolemy. The length of these segments or half-chords became associated with an angle through which the revolving line turned.[18]

The half-chord in Arabic is known as jiba and became confused with jaib. Arabic words were frequently written without vowels, and the consonants of both jiba and jaib are j and b.[19] Jaib, however, had nothing to do with the length of a half-chord since it meant 'the opening of a garment at the neck and bosom.' By the time European mathematicians became familiar with Arabic terms concerning half-chords, they were consistently calling half-chords by the term jaib with its meaningless reference to the bosom. Consequently, the European mathematicians translated jaib by the Latin sinus, meaning 'bosom' or 'fold'. The term sine for the half-chord is derived from the Latin sinus.[20]

While trigonometry was at first treated as a branch of astronomy, it was event-

13) Sarton, op. cit., Vol. II, Part I, p. 11.
14) Rom Landau, *Arab Contribution to Civilization* (San Francisco, The American Academy of Asian Studies, 1958), pp. 35-6.
15) Sarton, op. cit., Vol. II, Part I, p. 11.
16) Henry B. Fine, *Number System of Algebra* (New York, D.C. Heath and Company, 1890), p. 110.
17) Joseph Hell, *The Arab Civilization* (Lahore, Sh. Mohd. Ahmad, 1943), p. 96.
18) Dirk J. Struik, *A Concise History of Mathematics* (New York, Dover Publications, 1967), p. 74.
19) Indian Office Library, London, England, Arabic MSS, 772, fol: 17^b-18^a.

ually studied independently. The Muslims were vastly superior to the Greeks and the Indians in the area of trigonometry in so far as they enlarged or used tables of the six fundamental trigonometrical functions and established the funamental relations between them.[21]

III-3. Al-Battani

In the ninth century, just as he does today, man studied the mystery of God and the relationship between heaven and earth. Therefore, it is not surprising that the Muslims directed their attention to spherical trigonometry, and Al-Battani became their chief proponent.[22] Mohammed ibn Jabir ibn Sinan Abu Abu Abdullah Al-Battani was born in Battan, Mesopotamia in 850 and died in Damascus in 929 AD.[23] He was an Arabian prince, governor of Syria, and is considered the greatest Muslim astronomer and mathematician.[24]

Al-Battani is mainly responsible for the modern concepts and notations of trigonometrical functions and identities.[25] Various astrological writings, including a commentary on Ptolemy's Tetrabiblon ('four books'), are attributed to him, but his main work was an astronomical treatise with tables, De Scientia and De Numeris Stellarum et Motibus (About Science and Number of Stars and their Motion) which was extremely influential until the Renaissance. He made astronomical observations of remarkable range and accuracy throughout his life. His tables contain a catalog of fixed stars compiled during the year 800-1. He found that the longitude of the sun's apogee had increased 16 degrees and 47 minutes since Ptolemy's planetary theory, 150AD. That implies the discovery of the motion of the solar apsides. Al-Battani determined several astronomical coefficients with great accuracy; precession, 5415 seconds a year; inclination of the eliptic, 23 degrees and 35 minutes. He proved the possibility of annual eclipses of the sun, and he did not believe in the trepidation of the equinoxes.[26]

At that time, Al-Battani's work in

20) Hooper, *The River Mathematics*, op. cit., 224-5.
21) Rene Taton, *History of Science: Ancient and Medieval Science from the Beginning to 1450* (New York, Basic Books, 1963), Vol. I, pp. 410-11.
22) F.W. Kokomoor, 'The Status of Mathematics in India and Arabia During the "Dark Ages" of Europe', *The Mathematics Teacher*, XXIX (January 1936), 229.
23) Abi 'Abdullah Mohammed bin Sinan bin Jabir Al-Haruni, *Kitab Assij Assabi'* (Rome, Tubi 'a bi Madinat Rumiyah al-'Uzma, 1899), p. xi.
24) Stephan and Nandy Ronart, *Concise Encyclopaedia of Arabic Civilization: The Arab East* (New York, Frederick A. Praeger, 1960), p. xi.
25) Kokomoor, op. cit.

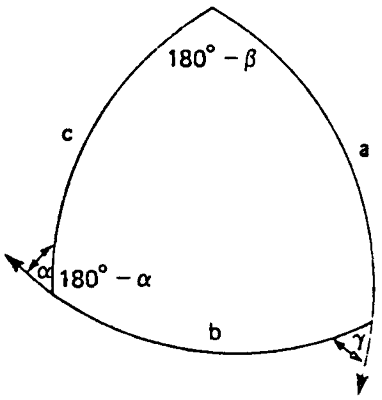

Fig. 5.1-A

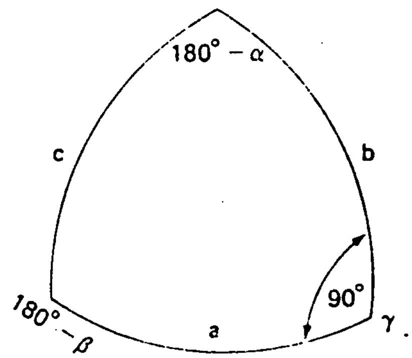

Fig. 5.1-B

astronomy was devoted mainly to trigonometry. He used sines regularly with a clear consciousness of their superiority over the Greek chords. Al-Battani, who was called 'Albategnius' by the Latins, completed the introduction of functions umbra extensa and umbra versa (cotangents and tangents) and gave a table of cotangents in terms of degrees.[27] He also knew the relation between the sides and angles for the general spherical triangle, which is expressed by the formula $\cos a = \cos b \cos c + \sin b \sin c \cos \alpha$, see Figure 5.1-A.[28] For the spherical right triangle with the right angle at C, he gave the formula, $\cos \beta - \cos b \sin \alpha$, see Figure 5.1-B.[29]

Al Battani not only computed sine, tangent, and cotangent tables from zero to 90 degrees with great accuracy, but he also applied algebraic operations to the trigonometric identity for the spherical triangle.[30] He developed the tables of cotangents based on the relation $\cot \alpha = \cos \alpha / \sin \alpha$.[31]

Al Battani also wrote such books as 'The book of the science of the ascensions of the zodiac in the spaces between the quadrants of the celestial sphere,' 'A letter on the exact determination of the

26) Bodleian Library, Oxford, England, Arabic MSS, 119, fol. (ff. 49^r-54^r).

27) George Sarton, *Introduction to the History of Science: From Homer to Omar Khayyam* (Baltimore, The Williams and Wilkins Company, 1953, Vol. I, pp. 602-3.

28) J.F. Scott, *A History of Mathematics: From Antiquity to the Beginning of the Nineteenth Century* (London, Taylor and Francis Ltd, 1969), p. 52.

29) Howard Eves, *An Introduction to the History of Mathematics* (New York, Holt, Rinehart, and Winston, 1969), 194.

30) George E. Reves, 'Outline of the History of Trigonometry,' *School Science and Mathematics*, LIII, No. 2 (February, 1953), p. 141.

31) Carra De Vaux, 'Astronomy and Mathematics,' *The Legacy of Islam* (London, Oxford University Press, 1931), p. 389.

quantities of the astrological applications,' and 'Commentary on Ptolemy's Tetrabiblon'. His principal work was al-Zij ('Astronomical Treatise and Tables'). This book contains the results of his observations and had a considerable influence not only on astronomy in the Muslim world but also on the development of astronomy and spherical trigonometry in Europe during the Middle Ages and the beginning of the Renaissance.[32]

Al-Battani wrote about the tangent, but its advantages apparently were not recognized by early Western scientists. In the thirteenth century many mathematicians referred to it as the umbra, and in the fourteenth century Levi Ben Gershon discussed the tangent in his De sinibus chords, et arcubus, item instrumento revelatore secretorum (about the sine, chords and arcs, also about instruments as yet being secrets), the first Western textbook of trigonometry. However. Regiomontanus, who was born in 1436 in Konigsberg, was one to appreciate the tangent's usefulness, which led him to revise the entire book after Al-Battani's writing.[33]

Called the Ptolemy of Baghdad, Al-Battani gave the rule for finding the altitude of the sun related to the height of a tower, L, and its shadow, x, by the formula:[34]

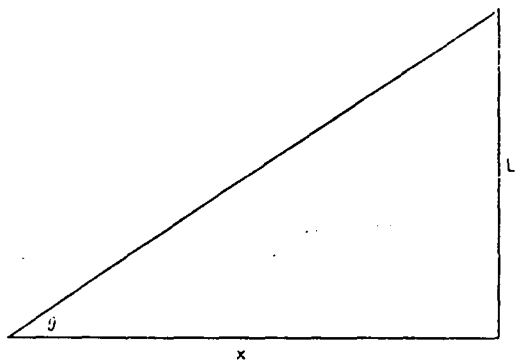

Fig. 5.2. Al-Battani's Plane Triangle

$$x=\frac{L \sin (90-\theta)}{\sin \theta}=L \cot \theta.$$

In discovering the motion of the sun's apogee, Al-Battani showed Ptolemy's error (of 17 degrees). By calculating the length of the year to be 365 days, 5 hours, 46 minutes, and 24 seconds, he was accurate to within two minutes of the exact time. He also corrected other observations of Ptolemy by establishing tables relevant to the motion of the sun, moon, and planets.[35]

III-4. Other Famous Muslim Mathematicians

The works of other Muslim mathematicians, Al-Biruni, Ibn Al-Shatir, Al-Khwarizmi, and Ibn Al-Haitham, also contri-

32) H.A.R. Gibb, J.H. Kramers, E. Levi-Provencal, and J. Schacht (eds.), *The Encyclopaedia of Islam* (London, Luzac and Company, 1960), New Edition, Vol. I, pp. 1104-5.

33) Rene Taton, *History of Science: The Beginnings of Modern Science* (New York, Basic Books, 1964), Vol. II, p. 17.

34) David Evgene Smith, *History of Mathematics* (New York, Ginn and Company, 1925), Vol. II, p. 608.

buted to the development of trigonometry. It will be noted that many of these mathematicians also made major developments in other areas of knowledge.

Al-Biruni

Al-Biruni was among those who laid the foundation for modern trigonometry.[36] As a philosopher, geographer, and astronomer,[37] Al-Biruni was not only a mathematician but a physicist as well. His contribution to physics was through studies in specific gravity and the origin of artesian wells.[38] Al-Biruni resided in India for nearly thirteen years(1017~30) and devoted himself to the study of the arts and sciences of the Hindus. He also had a remarkable knowledge of the Greek sciences and literature.[39]

Taki Ed Din al-Hilali considers Al-Biruni to be 'one of the very greatest scientists of all time.'[40] Six hundred years before Galileo, Al-Biruni had discussed the possibility of the earth's rotation around its own axis.[41]

Al-Biruni carried out geodesic measurements[42] and determined the magnitude of the earth's circumference in a most ingenious manner.[43] With the aid of mathematics, he fixed the direction to Mecca in mosques all over the world.[44]

Ibm Al-Shatir

'Ala al-Din 'Ali ibn Ibrahim Ibn Al-Shatir Al-Muwaqqit, an Arabian trigonometrist, lived from March 1306 until 1375 AD; he died in Damascus. His work progressed in Damascus where he was Muezzin at the great mosque Jami'al-Umawi.

Ibn Al-Shatir is considered one of the outstanding astronomers of his time. He made valuable astronomical observations, and wrote a special treatise, Rasd Ibn Shatir (Observatory of Ibn Shatir), conc-

35) J. Villin Marmery, *Progress of Science* (London, Chapman and Hall, 1895), pp. 33-4.
36) Mohammed Saffauri and Adnan Ifram (trans.), *Al-Biruni on Transits* (Beirut, American University of Beiurt Press, 1959), p. 17.
37) Sir William Cecil Dampier, *A Shorter History of Science* (New York, The Macmillan Company, 1945), p. 39.
38) Carl B. Boyer, *A History of Mathematics* (New York, John Wiley and Sons, 1968), pp. 263-4.
39) Bibhusan Datta and Avadhesh Narayan Singh, *History of Mathematics* (Lahore, Motilal Banarsi Das, 1935), Part I, p. 98.
40) Taki Ed Din al-Hilali *Die Einleitung Zu al-Birunis Steinbuch* (Leipzig, Otto Harrassowitz, 1941), p. vii.
41) Rom Landau, *The Arab Heritage of Western Civilization* (New York, Arab Information Center, 1962), p. 33.
42) Sir William Cecil Dampier, *History of Science* (New York, The Macmillan Company, 1943), p. 82.
43) C. Edward Sachau, *Chronologic Orientalischer Volker, Van al-Beruni*(Leipzig, In Commission Bei F.A. Brockhauş, 1878), pp. 184-6.
44) Jamil Ali (trans.), *Tahdid al-Amakin by al-Biruni* (Beirut, The American University Press, 1966), p. 8.

erning them. He devised astronomical instruments and wrote various treatises explaining their structure and use. With regular and precise observations, Ibn Al-Shatir investigated the motion of the celestial bodies and determined at Damascus the obliquity of the ecliptic to be 23 degrees 31 minutes in 1365 ; the correct value extrapolated from the present one is 23 degrees 31 minutes and 19.8 seconds.[45]

In Ibn Al-Shatir's book, a text of final inquiry in amending the elements, the Ptolemaic eccentric deferent was dispensed with completely and a second epicycle was introduced. Both the solar and lunar models are non-Ptolemaic, and what is of greatest interest is that the lunar theory is identical with that of Copernicus (1473~1543 AD) except for trivial differences in parameters.

Ptolemy assumed a circular path for the sun, but the orbit of Ibn Al-Shatir's sun deviated slightly from a circular motion. The major fault of the Ptolemaic lunar model is its exaggeration of the variation in lunar distance. The major Copernican contribution to the lunar theory consisted in the elimination of this Ptolemaic fault.[46]

There is no trace of the heliocentric concept in the treatise of Ibn Al-Shatir and Copernicus were compatible in their idea of utilizing only those celestial motions constructible by combinations of uniform circular motions.[47]

There is much similarity between the models of Ibnal-Shatir and those of Copernicus, both systems composed of constant length vectors rotating at a constant angular velocity. These astronomers abandoned the Ptolemaic equality; the lengths of corresponding vectors in the two systems are however, nearly equal, and are in many cases even identical.[48]

Al'Khwarizmi

The Caliph Al-Ma'mun built an observatory in Baghdad and another on the plains of Tadmor. His patronage stimulated astronomical observations of every kind. Tables of planetary motions were compiled, obliquity of the ecliptic was determined, and geodesic measurements were carefully made. Al-Khwarizmi was one of the first to compute astronomical

45) George Sarton, *Introduction to the History of Science* (Baltimore, The Williams and Wilkins Company, 1948), Vol. III, Part II, p. 1524.

46) Victor Roberts, 'The Solar and Lunar Theory of Ibm Al-Shatir, A Pre-Copernican Model,, *Isis*, XLVIII, Part 4, No. 154 (December 1957), p. 428.

47) E.S. Kennedy and Victor Roberts, 'The Planetary of Ibn al-Shatir,' *Isis*, L, Part 3, No. 161 (September 1959), 233.

48) Faud Abbud, 'The Planetary Theory of al-Shatir: Reduction of the Geometric Models to Numerical Tables,' *Isis*, LIII, Part 4, No. 174 (December 1962), 492.

and trigonometrical tables.[49] Included in Al-Khwarizmi's work in trigonometry were his one hundred tables of sines and cotangents values.[50]

Ibn Al-Haitham

Abu-'Ali al-Hasan ibn Al-Hasan ibn al-Haitham was born in Basrah, Iraq, in 965 AD and died in Cairo in 1039. He was one of the most important Muslim mathematicians and one of the greatest investigators of optics of all times. As a physician, he wrote commentaries on Aristotle and Galen.[51] His fame came from his treatise on optics which became known to Kepler during the seventeenth century.[52] This masterpiece, Kitab al-Manazir (Book of Mirrors), had a great influence on the training of later scientists in Western Europe.[53]

Ibn Al-Haitham's writings reveal his precise development of the experimental facilities. His tables of corresponding angles of incidence and refraction of light passing from one medium to another show how he nearly discovered the law of the ratio of sines for any given pair of media, later attributed to Snell. He investigated twilight relating it to atmospheric refraction by estimating the sun's height to be 19 degrees below the horizon at the commencement of the phenomenon in the mornings or at its termination in the evenings. The figure generally accepted now is 18 degrees.[54]

Ibn Al-Haitham estimated the height of the homogeneous atmosphere on this basis to be about 55 miles, a rather close approximation. He understood the laws of the formation of images in spherical and parabolic mirrors. He was also familar with the reasons for spherical aberration and of magnification produced by lenses. He gave a much more sound theory of vision than that of the Greeks, regarding the lens system of the eye itself to be the sensitive part. Ibn Al-Haitham was also able to solve a number of advanced questions in geometrical optics; for example, he solved the case of an aplanatic surface for reflection through his mastery of mathematics.[55] During his later years, Ibn Al-Haitham went to Egypt where he

49) Sarton, ***Introduction to the History of Science***, op. cit., Vol. I, p. 545.
50) Raymond Clare Archibald, 'Hindu, Arabic, and Persian Mathematics 600 to 1200', ***American Mathematics Monthly***, LVI (January 1949), 30).
51) Theodore F. Van Agenen, ***Beacon Lights of Science*** (New York, Thomas T. Crowell Company, 1924), pp. 45-6.
52) Solomon Bochner, ***The Role of Mathematics in the Rise of Science*** (Princeton, New Jersey, Princeton University Press, 1966), p. 304.
53) Van Wagnen, op. cit.
54) Nagula Shahin, 'Al-Daw'u al-Mustagtabu wa al-Tswiru al-Mghari al-Mulwan,' ***Gafilh Azzit***, XX (March-April 1972), pp. 7-8.

attempted to regulate the course of the Nile River. After working on this project for a time he earned his living by writing mathematical books.[56]

III-5. Summary

Trigonometry is a science satisfying two practical requirements. It inherits from both astronomy (science of celestial bodies) and from geometry (science of earth measurement) its main problem namely that of measuring an inaccessible distance. According to Edward J. Byng:

Trigonometry is mainly the original creation of the Arabians. So is analytical geometry, and algebra, whose very name is Arabic. The Arabs solved cubic equations by geometric construction. Following their revolutionary achievements in trigonometry, they invented celestial navigation, still the basis of a modern naval officer's training. Even 'our' terms, as used in modern navigation... 'azimuth, zenith, nadir'... are Arabic. The magnetic needle was discovered in China, but it was the Arabs who adopted it for navigation, inventing the mariners' compass. They also invented the astrolabe.[57]

Trigonometry depends upon mathematics, and equally vital are the related instruments for navigation. In that field, too, the Muslims proved to be the chief pioneers. During the Middle Ages, there were no telescopes, electrical gadgets, or radar; and measurements had to be made with purely mechanical instruments, such as the quadrant and the astrolabe. To reduce the margin of error, the Muslims made their instruments larger than any known previously. The most famous observatory at which these instruments were being used was at Maragha, in the thirteenth century, where distinguished astronomers from many countries collaborated.

The Muslims were also acquainted with the elements of spherical trigonometry, associated with Al-Battani, the earliest of the many distinguished Muslim astronomers.

The invaluable contributions of the Muslim mathematicians to trigonometry overshadow their work in the field of geometry. Although they did not extend the theory of geometry, the Muslims did establish a close relationship between geometry and algebra in their geometrical solutions of algebraic problems. Chief among their contributions was the translation of Euclid's Elements from Greek to Arabic.

55) Ibid.

56) M. Th. Houtsma, A.J. Wensinck, T.W. Arnold, W. Heffening, and E. Levi-Provencal (eds.), *The Encyclopaedia of Islam* (London, Luzac and Company, 1927), Vol. II, p. 382,

* Astrolabc is the predecessor of the Modern sextent,