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An Analytic Model of the M_2 Tide near Cheju Island

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濟州島 周邊 M_2潮에 대한 解析的 모델

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Abstract: The M_2 tide near Cheju Island is studied on the basis of a scattering of incident tidal wave by an elliptic island in an ocean of constant depth. The amplitudes of incident and scattered waves are determined from the required boundary condition and by the least squares fit of the model to the tidal observations at 7 stations in the island. The tidal chart for M_2 tide near the island is analytically constructed.

要約: 타원형 섬에 의한 조석파의 산란 이론에 근거하여, 제주도 주변의 M_2 반일주조 조석을 구명하였다. 제주도를 수심이 일정한 해양 가운데 놓여있는 타원형의 섬으로 근사화한 후, 입사파와 산란파의 진폭을 연안 경계조건과 7개 검조소의 조화상수에 대한 최소자승법을 통하여 결정하였다. 이와 같은 이론식에 근거하여 제주도 주변 M_2조에 대한 조석도를 해석적으로 작성하였다.

INTRODUCTION

Cheju Island is elliptical in shape, located to the south of the Korean Peninsula. The tidal chart in the Yellow and East China seas by Ogura (1933), shown in Fig. 1, suggests that the M_2 semi-diurnal tide near Cheju Island can be regarded basically as a plane wave propagating northwest with a wavelength of about 1000km. The general features of tidal motion in the Yellow and East China seas are rather well understood by the tidal record (Ogura, 1933), by hydro-numerical model (An, 1977; Choi, 1980), and also by an analytic investigation (Kang, 1984). However, a detailed information on the tide near Cheju Island cannot be inferred from those studies. The main objective of this study is to understand the 'local' behaviour of tidal waves in the vicinity of Cheju Island.

The effects of an elliptic island on tidal waves were first investigated by Proudman (1914, 1925), whose solution, given in terms of stream function and velocity potential, is valid when the horizontal dimension of the area considered is much smaller than the wavelength of incident wave. Reynolds (1975) developed a numerical model for tidal motions disturbed by an elliptic ridge. Larsen (1977) made tidal charts in the Pacific Ocean near Hawaii considering deformations of free and forced tidal waves by the Hawaiian Ridge.

In this paper the tide near Cheju Island is assumed to consist of an incident plane wave, so called Sverdrup wave, and another wave scattered by the island. The amplitude of the resultant tidal wave is determined by the least squares fit of the model to the harmonic constants at 7 tidal stations around Cheju Island (Fig. 2). Using this simple analytic model, tidal chart for the M_2 semi-diurnal tide near Cheju Island is analytically constructed. The M_2 tide only is considered here, because it is the strongest tidal component.

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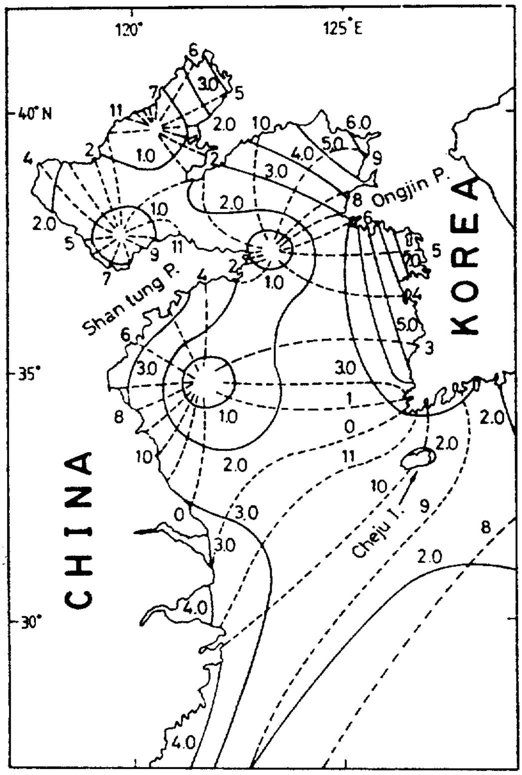

Fig. 1. Tidal chart in the Yellow and the East China Seas (after Ogura, 1933). Co-tidal lines of M_2 tide referred to 135°E and spring co-range lines, 2 (M_2+S_2), in meters.

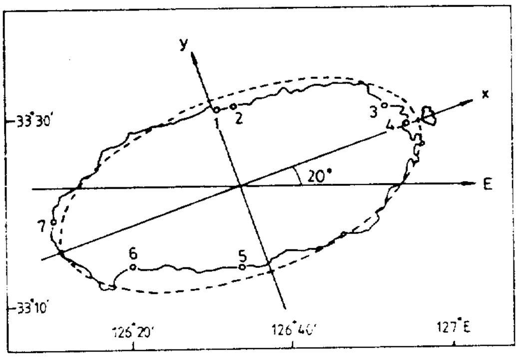

Fig. 2. Elliptical approximation of Cheju Island. The major and minor axes are 36.6 and 17.6km, respectively, and the major axis is tilted 20° with respect to the east.

MODEL

1. Governing equations

The linearized momentum and continuity equations for barotropic motion in an f-plane ocean of constant depth are

$$\frac{\partial u}{\partial t}-fv=-g\frac{\partial \zeta}{\partial x}, \quad \frac{\partial v}{\partial t}+fu=-g\frac{\partial \zeta}{\partial y}$$

$$\frac{\partial \zeta}{\partial t}+h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0, \qquad (1.1)$$

where u and v are velocity components to the x-and y-directions, respectively, t the time, g the gravity, ζ the sea-level elevation, and h the depth.

For an analytic investigation of tidal waves near an elliptic island, it is convinient to formulate the problem in elliptic coordinates (ξ, η), defined in terms of the cartesian coordinates (x, y) as $x=c\cosh\epsilon\cos\eta$, $y=c\sinh\xi\sin\eta$, where c is the focus distance from the origin, and both ξ and η are in radians ($\xi>0$, $0\leqslant\eta\leqslant 2\pi$). The curves of constant ξ are ellipses and curves of constant η are hyperbolas (Fig. 3). An elliptic island with the major and minor axes of a and b, respectively, is described by $x^2/a^2+y^2/b^2=1$ in the cartesian coordinates, and the focus distance c is given by $c=\sqrt{a^2-b^2}$. In elliptic coordinates the island is described by $\xi=\xi_0$, where ξ_0 is given by $\log[(a+b)/c]$.

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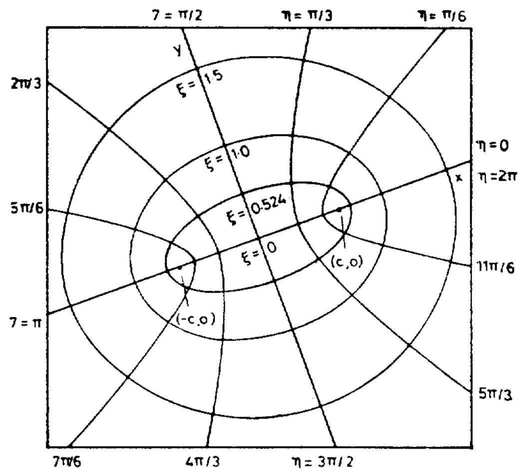

Fig. 3. Elliptic coordinates (ξ, η) with a confocal distance c of 32.1km. The square is 160×160km², and Cheju Island is represented by $\xi=0.524$.

1), can be written in elliptic coordinates as

$$\frac{\partial u'}{\partial t}-fv'=-\frac{g}{s}\frac{\partial\zeta}{\partial\xi}$$

$$\frac{\partial v'}{\partial t}+fu'=-\frac{g}{s}\frac{\partial\zeta}{\partial\eta} \qquad (1.2)$$

$$\frac{\partial\zeta}{\partial t}+\frac{1}{s^2}\left[\frac{\partial}{\partial\xi}(su')+\frac{\partial}{\partial\eta}(sv')\right]=0, \qquad (1.3)$$

where u' and v' are velocity components to the ξ- and η-directions, respectively, and

$$s=c\sqrt{\sinh^2\xi+\sin^2\eta}.$$

We assume time-harmonic motion such that

$$u'(\xi,\eta,t)=\mathrm{Re}[U(\xi,\eta)\ \exp(-i\omega t)]$$

$$v'(\xi,\eta,t)=\mathrm{Re}[V(\xi,\eta)\ \exp(-i\omega t)]$$

$$\zeta(\xi,\eta,t)=\mathrm{Re}[Z(\xi,\eta)\ \exp(-i\omega t)], \qquad (1.4)$$

where U, V and Z are complex amplitudes, and ω is the angular frequency of the forced tide. If we substitute (1.4) into (1.2) and solve for U and V, we obtain

$$U=\frac{g}{s(\omega^2-f^2)}\left[-i\omega\frac{\partial Z}{\partial\xi}+f\frac{\partial Z}{\partial\eta}\right] \qquad (1.5)$$

$$V=\frac{g}{s(\omega^2-f^2)}\left[-i\omega\frac{\partial Z}{\partial\eta}-f\frac{\partial Z}{\partial\xi}\right] \qquad (1.6)$$

The continuity equation (1.3) also can be simplified by using the time-harmonic assumption (1.4). If we substitute (1.5) and (1.6) into this simplified continuity equation and rearrange we obtain the Helmholtz equation for Z alone,

$$\frac{\partial^2 Z}{\partial\xi^2}+\frac{\partial^2 Z}{\partial\eta^2}+s^2k^2Z=0, \qquad (1.7)$$

where k is the wavenumber of plane wave ($k=\frac{\sqrt{\omega^2-f^2}}{gh}$), so called Sverdrup wave (Platzman, 1971).

The required boundary condition of no normal flow ($U=0$) across the coast of an elliptic island is, from (1.5),

$$-i\omega\frac{\partial Z}{\partial\xi}+f\frac{\partial Z}{\partial\eta}=0 \text{ at } \xi=\xi_0. \qquad (1.8)$$

The sea-level elevation should satisfy (1.7) and (1.8).

2. Solution near an elliptic island

Over a region of linear dimension r such that $kr\ll 1$, the Helmholtz equation (1.7) can be approximated to a Laplace equation

$$\frac{\partial^2 Z}{\partial\xi^2}+\frac{\partial^2 Z}{\partial\eta^2}=0. \qquad (2.1)$$

The incident plane wave solution Z_0 satisfying (3.1) is

$$Z_0=A(1+ikr), \qquad (2.2)$$

where A is a complex amplitude and

$$r=x\cos\alpha+y\sin\alpha$$
$$=c(\cosh\xi\cos\eta\cos\alpha+\sinh\xi\sin\eta\sin\alpha),$$

α the direction of wave propagation measured counterclockwise from the x-axis. Note that the Sverdrup wave solution of (1.7), viz.,

$$Z=A\exp(ikr) \qquad (2.3)$$

approaches to (2.2) if $kr\ll 1$. This suggests that the Laplace equation (2.1) is a valid approximation of (1.7) only when $kr\ll 1$. This condition is reasonably satisfied in the vicinity of Cheju Island because the major axis of Cheju Island is only 36.6km and wavelength is about 1000km.

In order to satisfy the boundary condition (1.8) we need another wave scattered by an elliptic island. The scattered wave solution Z_s of (2.1), which decays with distance from the island, is

$$Z_s=A\exp(-\xi)\ (B\sin\eta+C\cos\eta). \qquad (2.4)$$

The resultant sea-level Z is obtained by superposing Z_0 and Z_s as

$$Z(\xi,\eta)=A[1+ikc\ (\cosh\xi\cos\eta\cos\alpha+\sinh\xi\sin\eta\sin\alpha)+\exp(-\xi)\ (B\sin\eta+C\cos\eta)]. \qquad (2.5)$$

The constants B and C can be determined by substituting (2.5) into (1.8) and collecting coefficients of $\sin\eta$ and $\cos\eta$ separately, i.e.,

$$fB-i\omega C=kc\exp(\xi_0)\sinh\xi_0\ (\omega\cos\alpha-if\sin\alpha)$$

$$i\omega B+fC=-kc\exp(\xi_0)\cosh\xi_0\ (\omega\sin\alpha-if\cos\alpha). \qquad (2.6)$$

From these two simultaneous linear equations we get B and C. Finally, using (2.6) and manuplating with hyperbolic identities, (2.5) becomes

$$Z(\xi,\eta)=A\{1+\beta\ \exp(\xi_0-\xi)\ f\omega\sin(\alpha-\eta)+i\beta[\omega^2\cosh(\xi_0-\xi)+f^2\sinh(\xi_0-\xi)]\cos(\alpha-\eta)\},\tag{2.7}$$

where $\beta=\dfrac{k^2(a+b)}{\omega^2-f^2}$

The sea-level at the coast of an elliptic island ($\xi=\xi_0$) is, from (3.7),

$$Z(\xi_0,\eta)=A[1+\beta f\omega\sin(\alpha-\eta)+i\beta\omega^2\cos(\alpha-\eta)].\tag{2.8}$$

The complex amplitude A can be determined from the harmonic constant at the coast, as will be shown in the next section.

3. Determination of amplitude

The amplitude and phase of M_2 tide at 7 stations in Cheju Island (Fig. 2), taken from Choi (1980), are listed in Table 1. These values enable us to determine the amplitude A.

Table 1. Harmonic constants of the M_2 tide in Cheju Island. For location see Fig. 2.

	Amp. H(cm)	Phase φ	Elliptic η(rad)
1. Cheju	71.3	304°	0.473π
2. Whabukri	66	301°	0.445π
3. Sewha	61.2	284°	0.171π
4. Udo W.C.	66	274°	0.023π
5. Seogwipo	77	271°	1.473π
6. Whasun	77.4	284°	1.189π
7. Chagui I.	66	301°	0.920π

The sea-level associated with the model at the j-th station Z_j is

$$Z_j(\xi_0,\eta_j)=(A_r+iA_i)[1+\beta f\omega\sin(\alpha-\eta_j)+i\beta\omega^2\cos(\alpha-\eta_j)],\tag{3.1}$$

where A_r and A_i are the real and imaginary parts of A, respectively. The A can be determined by the least squares fit of the model to the observations. We minimize a function E defined by

$$E=\sum_{j=1}^{7}|Z_j-H_j\exp(i\varphi_j)|^2,\tag{3.2}$$

where H_j and φ_j are respectively the amplitude and phase of the M_2 tide at the i-th station (Table 1). The minimization conditions, $\partial E/\partial A_r=0$ and $\partial E/\partial A_i=0$, yield

$$A_rD=\sum_j H_j\cos\varphi_j[1+\beta f\omega\sin(\alpha-\eta_j)]+\sum_j H_j\beta\omega^2\sin\varphi_j\sin(\alpha-\eta_j)$$

$$A_iD=\sum_j H_j\sin\varphi_j[1+\beta f\omega\sin(\alpha-\eta_j)]-\sum_j H_j\beta\omega^2\cos\varphi_j\cos(\alpha-\eta_j),\tag{3.3}$$

where $D=\sum_j[1+\beta f\omega\sin(\alpha-\eta_j)]^2+\sum_j[\beta\omega^2\cos(\alpha-\eta_j)]^2$.

I approximate Cheju Island as an elliptic island located at 33°N and oriented 20° from the east (see Fig. 2). The major and minor axes of the island are 36.6 and 17.6km, respectively. I assume that the undisturbed plane wave with a wavelength of 1000km propagates 140° referred to the east, *i.e.* $\alpha=120°$. The A for the M_2 tide determined by (3.3) is $A=18.0-58.7i$(cm).

4. Tidal chart

The tidal chart of the M_2 tide near Cheju Island can be constructed as follows. The sea-level elevation can be represented as

$$\zeta(\xi,\eta,t)=Z(\xi,\eta)\ \exp(-i\omega t)=|Z(\xi,\eta)|\exp[i(\theta-\omega t)],\tag{4.1}$$

where $|Z(\xi,\eta)|=\sqrt{Z_r^2+Z_i^2}$,

$$\theta(\xi,\eta)=\tan^{-1}(Z_i/Z_r),$$

and Z_r and Z_i are respectively the real and imaginary parts of $Z(\xi,\eta)$ given by (2.7). Co-range lines are constructed following the lines along which $|Z(\xi,\eta)|$ is constant. Co-tidal lines are described by curves along which $\theta(\xi,\eta)-\omega t=$constant, or $t=\dfrac{\theta(\xi,\eta)}{\omega}+$const. Fig. 4 shows the co-tidal chart of the M_2 tide, in degrees referred to 135°E, in the 160 by 160 km area surrounding Cheju Island. The co-range chart in the same area is shown in Fig. 5.

DISCUSSION AND CONCLUSION

In this paper, the tidal chart of the M_2 tide near Cheju Island is analytically constructed by superposition of an incident plane wave and another wave scattered by the island. The direction and wavelength of the incident wave are inferred from the tidal chart of the Yellow and the East China seas by Ogura (1933). The amplitude of tidal wave is determined by the least squares fitting of the model to the observations at 7 tidal stations in Cheju Island.

For a mathemtical simplicity, various approximations are used in the model. Cheju Island is approximated as an elliptic island surrounded by a vertical wall in an ocean of constant depth. The influence of the Korean Peninsula on the incident wave is neglected.

Because of these approximations, the analytic tidal chart of this paper does not agree exactly with the reality. However, the model gives us insights on the local behaviour of tidal waves near Cheju Island. The analytic result of phase (Fig. 4) agrees reasonably with the observations at 7 costal stations (Table 1). The phase difference between the model and the observation never exceed 6°. The analytic result of amplitude distribution (Fig. 5), however, does not match well with the observations. The average difference in the amplitude between the model and the observation at 7 coastal stations is

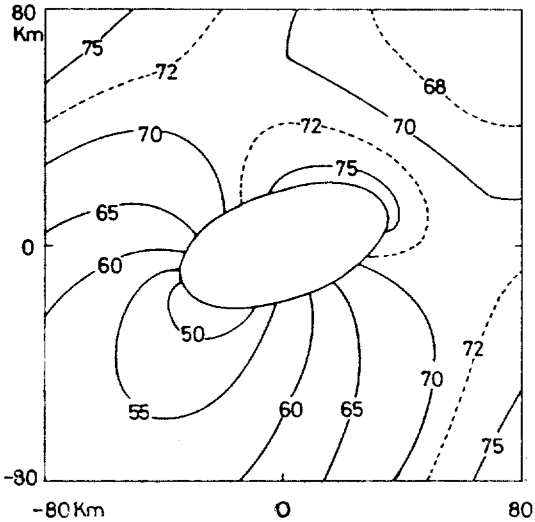

Fig. 5. Co-range chart around Cheju Island, amplitude in cm.

12cm. This difference arises, perhaps, due to a deviation of the coastal geometry from an ellipse, a non-uniform distribution of bottom topography, and scattering of tidal wave by the southern coast of Korean Peninsula. The model in this paper works successfully for the semi-diurnal tide, but cannot be applied for the diurnal tide, the period of which is longer than an inertial period of 22 hours at 33°N.

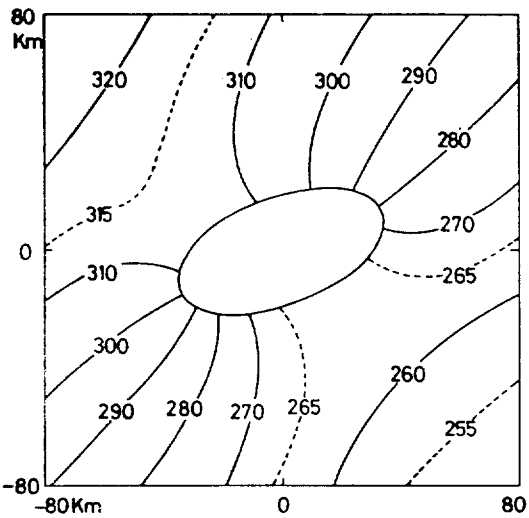

Fig. 4. Co-tidal chart around Cheju Island in degrees referred to 135°E.

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