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Properties of Detection Matrix and Parallel Flats Fraction for 3^n Search Design+

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ABSTRRACT

A parallel flats fraction for the 3^n design is defined as union of flats $\{\underset{\sim}{t}|A\underset{\sim}{t}=\underset{\sim}{c}_i \pmod 3\}$, $i=1, 2, \cdots, f$ and is symbolically written as $A\underset{\sim}{t}=C$ where A is rank r. The A matrix partitions the effects into $u+1$ alias sets where $u=(3^{n-r}-1)/2$. For each alias set the f flats produce an $ACPM$ from which a detection matrix is constructed. The set of all possible parallel flats fraction C can be partitioned into equivalence classes. In this paper, we develop some properties of a detection matrix and C.

1. Introduction

A paralled flats fraction for the 3^n factorial experiment is defined as the union of flats $\{\underset{\sim}{t}|A\underset{\sim}{t}=\underset{\sim}{C}_i \pmod 3),\ i=1,2,\cdots, f\}$ and is symbolically written as $A\underset{\sim}{t}=C$ where A is a rxn matrix with rank r and $C=(\underset{\sim}{C}_1, \underset{\sim}{C}_2, \cdots, \underset{\sim}{C}_f)$ is a rxf matrix. Note that f denotes the number of flats.

The A matrix partitions the effects into $u+1$ alias sets where $u=(3^{n-r}-1)/2$. For each alias set the f flats produce an alias component permutation matrix($ACPM$) with elements from the permutation group S_3.

Um(1981) showed that the set of all possible parallel flats fraction C for a given A and given size can be partitioned into equivalence classes. Table 1 shows the equivalence classes of C matrix for the 3^4 factorial.

A detection vector of the $ACPM$ was constructed for each combination of k or fewer two-factor interactions by Um(1983). Also the relationship between the detection vectors

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has been shown. Table 2 shows the detection matrix for the 3^4 factorial.

2. Basic Lemmas

Suppose that the design T is obtained from solution to $A\underline{t}=C$, where A is rxn with rank r and C is rxf matrix, Then the following lemmas about the C matrix can be summarized from Um(1981).

Lemma 1. Let the design T^* be obtained from solutions to $A\underline{t}=C^*$ where C^* is obtained from C by permuting columns of C except the first column. Then the designs T and T^* are equivalent and C, C^* belong to the same equivalence class.

Lemma 2. Let the design T^{**} be obtained from solutions to $A\underline{t}=C^{**}$ where C^{**} is obtained by adding the vector $\underline{v}$ with elements in $GF(3)$ to each of the columns of C. Choose $\underline{v}$ such that there exists one column of 0's after adding $\underline{v}$ to each column of C, then the designs T and T^{**} are equivalent and C, C^{**} belong to the same equivalence class.

Lemma 3. Let the design T^{***} be obtained from solutions to $A\underline{t}=C^{***}$ where $C^{***}=2C$. Then the designs T and T^{***} are equivalent and C, C^{***} belong to the same equivalence class.

Lemma 2 and Lemma 3 can be combined to establish designs which are equivalent. If T^* is obtained from solutions to $A\underline{t}=C^*$ where $C^*=2C_+(\underline{v}, \underline{v}, \cdots, \underline{v})$, then the designs T and T^* are equivalent and C, C^* belong to the same equivalence class.

3. Main Results

Note that elements of $ACPM$ depend on a C matrix and the detection vectors are obtained from $ACPM$. It is important to relate the detection vectors to the C matrix. We now develop some relationships between the equivalence class of C matrix and the detection vectors.

Lemma 4. Let the matrix C^* be obtained from C by permuting columns of C except the first column. Then the detection vector obtained from C^* are just a permutation of elements of the detection vectors obtained from C.

Proof. Each column of C matrix corresponds to one row of $ACPM$ P_i, $i=1, 2, \cdots, u$, where $u+1$ is the number of alias sets.

Suppose that two columns of C, say column 2 and column 3, are permuted. Then for every i the corresponding rows of Pi, that is row 2 and row 3, are interchanged. Therefore, the column1-2 (the difference between row 1 and row 2) of the detection vector obtained from C becomes the columm 1-3 (the difference between row 1 and row 3) of the detection vector obtained from C^*. This is true for any permuting columns of C except the first column. This completes the proof.

Lemma 5. Two detection vectors obtained from C and C^{**}, where $C^{**}=2C$, are the same.

Proof. In order to get $ACPM$, suppose that we have

$$P^*=(0,\ x_2\ \underline{t}_0'\ \underline{e}_2,\ \ x_3\underline{t}_0'\ \underline{e}_3, \cdots,\ x_q\underline{t}_0'\ \underline{e}_q)\text{(See Um(1980)).}$$

Then the $ACPM$ whose elements are composed of 0, 1 and 2 can be obtained. Therefore, multiplying the matrix C by 2 implies simply that the elements of $ACPM$ are multiplied by 2. Afer this the transformations are performed. Then (012) obtained from C becomes (021) obtained from C^{**}, and (021) becomes (012). Hence the detection vectors are not affected. This completes the proof.

The implication of Lemma 5 is that if one column of C^{**} can be obtained by multiplying the corresponding column of C by 2 then the detection elements for the difference between the first row and the corresponding row of $ACPM$ are the same for C and C^{**}.

Lemma 6. Let the matrix C^{***} be obtained from C by adding nonzero vector $\underline{v}$. Then the detection vector obtained from C^{***} are the same with the detection vector obtained from C or a permutation of columns of the detection vector obtained from C.

Proof. Suppose that we choose $\underline{v}$ such that the second column will have $\underline{0}$ after adding $\underline{v}$ to the C. Then the first column of C^{***} is $\underline{0}$ and the second column is $\underline{v}$. This implies that the second column of C^{***} can be obtained by multiplying the second column of C by 2. Therefore, the detection elements for the difference between the first row and the second row of $ACPM$ are not changed with C and C^{***}. This means that for any choice of column of C the detection element for the difference the first row and the corresponding row of $ACPM$ are the same for C and C^{***}.

Without lose of generality let $P^*=(0,\ c_1,\ c_2)$ where the corresponding effects are E_1, E_2, E_3. Let $C=\begin{bmatrix} 0 & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \end{bmatrix}$ where $c_{ij} \in GF(3)$ and the columns are different from each other, and let $V'=(-c_{12}\ -c_{22})$. Consider the detection vectors for various values of c_{ij}.

(Case 1). One of c_{12} and c_{22} is zero. Suppose that $c_{22}=0$. Thenclearly c_{12} is 1 or 2,

$C^{***}=\begin{bmatrix}0 & -c_{12} & -c_{12}+c_{13}\\ 0 & 0 & c_{23}\end{bmatrix}$, and the following $ACPM$ are obtained from P^* :

$ACPM$ for C

$$\begin{matrix}E_1 & E_2 & E_3\end{matrix}\\ \begin{bmatrix}0 & 0 & 0\\ 0 & c_{12} & 0\\ 0 & c_{13} & c_{23}\end{bmatrix}$$

$ACPM$ for C^{***}

$$\begin{matrix}E_1 & E_2 & E_3\end{matrix}\\ \begin{bmatrix}0 & 0 & 0\\ 0 & -c_{12} & 0\\ 0 & -c_{12}+c_{13} & c_{23}\end{bmatrix}$$

For any choice of c_{23} the detection vector for the effect E_3 is the same with C and C^{***}.

Consider the detection vectors for the effect E_2. Suppose that $c_{12}=c_{13}$. Then $-c_{12}\doteqdot 0$ and $-c_{12}+c_{13}=0$.

Therefore, the detection vector for C^{***} can be obtained by interchanging the second column with the third column of the detection vector for C. Suppose that $c_{12}\doteqdot c_{13}$. Then theare are four possible cases:

$(c_{12}, c_{13})=(1, 0), (2, 0), (1, 2), (2, 1)$.

a) Let $c_{12}=1$ and $c_{13}=0$. Thne $C^{***}=\begin{bmatrix}0 & 2 & 2\\ 0 & 0 & c_{23}\end{bmatrix}$ and

$ACPM$ for C

$$\begin{matrix}E_1 & E_2 & E_3\end{matrix}\\ \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & c_{25}\end{bmatrix}$$

$ACPM$ for C^{***}

$$\begin{matrix}E_1 & E_2 & E_3\end{matrix}\\ \begin{bmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 2 & c_{23}\end{bmatrix}$$

The detection vectors for C^{***} can be obtained by permuting the second column and the third column of the detection vectors for C. Sinmilarly, this holds for $c_{12}=2$ and $c_{13}=0$.

b). Let $c_{12}=1$ and $c_{13}=2$. Then we have

$ACPM$ for C

$$\begin{matrix}E_1 & E_2 & E_3\end{matrix}\\ \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2 & c_{23}\end{bmatrix}$$

$ACPM$ for C^{***}

$$\begin{matrix}E_1 & E_2 & E_3\end{matrix}\\ \begin{bmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 1 & c_{23}\end{bmatrix}$$

Both matrices produce the same detection vectors. This holds for $c_{12}=2$ and $c_{13}=1$. Similar arguments hold for $c_{12}=0$.

(Case 2). c_{12} and c_{22} are 1 or 2. Suppose that $c_{12}=c_{22}$. Then we have

$ACPM$ for C

$$\begin{matrix}E_1 & E_2 & E_3\end{matrix}\\ \begin{bmatrix}0 & 0 & 0\\ 0 & c_{12} & c_{12}\\ 0 & c_{13} & c_{23}\end{bmatrix}$$

$ACPM$ for C^{***}

$$\begin{matrix}E_1 & E_2 & E_3\end{matrix}\\ \begin{bmatrix}0 & 0 & 0\\ 0 & -c_{12} & -c_{12}\\ 0 & -c_{12}+c_{13} & -c_{12}+c_{23}\end{bmatrix}$$

It is clear that if $c_{13}=0$ or 1 then for the effect E_2 the detection vector for C^{***} can be obtained by permuting the second column and the third column of the detection vectors for C. If $c_{13}=2$ then the detection vectors are the same for both cases.

Suppose that $c_{12}\neq c_{22}$. Then we have

$$ACPM \text{ for } C \quad \begin{matrix} E_1 & E_2 & E_3 \end{matrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_{12} & c_{22} \\ 0 & c_{13} & c_{23} \end{bmatrix} \qquad ACPM \text{ for } C^{***} \quad \begin{matrix} E_1 & E_2 & E_3 \end{matrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & -c_{12} & -c_{22} \\ 0 & -c_{12}+c_{13} & -c_{22}+c_{23} \end{bmatrix}$$

Obviously, if $c_{13}=0$ or 1, then for the effect E_2 the detection vector for C^{***} can be obtained by permuting the two columns of detection vector for C. If $c_{13}=2$ then the detection vectors are the same for both cases. The same argument holds for the effect E_3 with the various values of c_{23}.

The above arguments in Case 1 and Case 2 are true for any choice of $\underline{v}$ and for any form of P^*.

This completes the proof.

Combining Lemmas 4, 5 and 6the following theorem is obtained.

Theorem. Suppose that C matrix C_1 and C_2 are rxf matrices where C_1 and C_2 belong to the same equivalence class. Then the detection vectors for C_1 and C_2 are the same or permute each other.

4. Example

Consider a 3^4 factorial experiment for which it can be assumed that all three and four-factor interaction effects are negligible. The A matrix for this example will be taken as

$$A=\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix},$$

thus there are flats of size nine. The alias sets are

$S_0=\{\mu\}$,

$S_1=\{F_1,\ F_2,\ F_3,\ F_2F_4^2,\ F_3F_4\}$, $S_2=\{F_2,\ F_1F_3,\ F_1F_4,\ F_3F_4^2\}$,

$S_3=\{F_3,\ F_1F_2,\ F_1F_4^2,\ F_2F_4\}$, $S_4=\{F_4,\ F_1F_2^2,\ F_1F_3^2,\ F_2F_3^2\}$.

An example of a parallel flats fraction in 27 runs is given with

$C=(\underline{C}_1,\ \underline{C}_2,\ C_3)$ as $C=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

By choosing the main effect in each alias set as the identified effect, the $ACPM$ are

$$
\begin{array}{cccc} F_1 & F_2F_3 & F_2F_4^2 & F_3F^4 \end{array}
\qquad
\begin{array}{cccc} F_2 & F_1F_3 & F_1F_4 & F_3F_4^2 \end{array}
$$

$$
P_1=\begin{bmatrix} e & e & e & e \\ e & e & (021) & (012) \\ e & (021) & (012) & e \end{bmatrix}
\qquad
P_2=\begin{bmatrix} e & e & e & e \\ e & e & (012) & (021) \\ e & (021) & (021) & (021) \end{bmatrix}
$$

$$
\begin{array}{cccc} F_3 & F_1F_2 & F_1F_4^2 & F_2F_4 \end{array}
\qquad
\begin{array}{cccc} F_4 & F_1F_2^2 & F_1F_3^2 & F_2F_3^2 \end{array}
$$

$$
P_3=\begin{bmatrix} e & e & e & e \\ e & e & (021) & (012) \\ e & (021) & e & (012) \end{bmatrix}
\qquad
P_4=\begin{bmatrix} e & e & e & e \\ e & (021) & (021) & (021) \\ e & (012) & e & (021) \end{bmatrix}
$$

Table 1 shows the equivalence classes of C and Table 2 shows the detection vectors for this example.

TABLE 1. EQUIVALENCE CLASSES OF 0 MATRIX FOR 3^4, DESIGN

CLASS 1					
0 0 1	0 0 1	0 2 2	0 1 0	0 1 0	0 2 2
0 1 2	0 2 1	0 2 1	0 2 1	0 1 2	0 1 2
0 0 2	0 0 2	0 1 1	0 2 0	0 2 0	0 1 1
0 2 1	0 1 2	0 1 2	0 1 2	0 2 1	0 2 1
CLASS 2					
0 1 2	0 2 1	0 2 1	0 2 1	0 1 2	0 1 2
0 0 1	0 0 1	0 2 2	0 1 0	0 1 0	0 2 2
0 2 1	0 1 2	0 1 2	0 1 2	0 2 1	0 2 1
0 0 2	0 0 2	0 1 1	0 2 0	0 2 0	0 1 1
CLASS 3					
0 0 1	0 0 1	0 2 2	0 1 0	0 1 0	0 2 2
0 1 0	0 2 2	0 1 0	0 0 1	0 2 2	0 0 1
0 0 2	0 0 2	0 1 1	0 2 0	0 2 0	0 1 1
0 2 0	0 1 1	0 2 0	0 0 2	0 1 1	0 0 2
CLASS 4					
0 0 1	0 0 1	0 2 2	0 1 0	0 1 0	0 2 2
0 1 1	0 2 0	0 0 2	0 1 1	0 0 2	0 2 0
0 0 2	0 0 2	0 1 1	0 2 0	0 2 0	0 1 1
0 2 2	0 1 0	0 0 1	0 2 2	0 0 1	0 1 0
CLASS 5					
0 0 0	0 0 0				
0 1 2	0 2 1				
CLASS 6					
0 1 2	0 2 1				
0 0 0	0 0 0				
CLASS 7					
0 1 2	0 2 1				
0 1 2	0 2 1				
CLASS 8					
0 1 2	0 2 1				
0 2 1	0 1 2				

TABLE 2. THE DETECTION MATRIX FOR THE 3^4 DESIGN

			P1			*P2*			*P3*			*P4*		
			1-2	1-3	2-3	1-2	1-3	2-3	1-2	1-3	2-3	1-2	1-3	2-3
1	MAIN		0	0	0	0	0	0	0	0	0	0	0	0
2		12	0	0	0	0	0	0	0	1	1	1	1	1
3		13	0	0	0	0	1	1	0	0	0	1	0	1
4		14	0	0	0	1	1	1	1	0	1	0	0	0
5		23	0	1	1	0	0	0	0	0	0	1	1	0
6		24	1	1	1	0	0	0	1	1	0	0	0	0
7		34	1	0	1	1	1	0	0	0	0	0	0	0
8	12	13	0	0	0	0	1	1	0	1	1	1	1	1
9	12	14	0	0	0	1	1	1	1	1	1	1	1	1
10	12	23	0	1	1	0	0	0	0	1	1	1	1	1
11	12	22	1	1	1	0	0	0	1	1	1	1	1	1
12	12	34	1	0	1	1	1	0	0	1	1	1	1	1
13	13	14	0	0	0	1	1	1	1	0	1	1	0	1
14	13	23	0	1	1	0	1	1	0	0	0	1	1	1
15	13	24	1	1	1	0	1	1	1	1	0	1	0	1
16	13	34	1	0	1	1	1	1	0	0	0	1	0	1
17	14	23	0	1	1	1	1	1	1	0	1	1	1	0
18	14	24	1	1	1	1	1	1	1	1	1	0	0	0
19	14	34	1	0	1	1	1	1	1	0	1	0	0	0
20	23	24	1	1	1	0	0	0	1	1	0	1	1	0
21	23	34	1	1	1	1	1	0	0	0	0	1	1	0
22	24	34	1	1	1	1	1	0	1	1	0	0	0	0

COLUMN 4-8 DENOTE THE SUBSRIPTS OF TWO-FACTOR INTERACTIONS

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