〈Research Paper〉

Journal of the Korean
Statistical Society
Vol. 13, No. 1, 1984

Testing Hypothesis for the Logistic Model with Estimated Parameters: Modified Tables of Critical Values for K-S Type Statistic

Chong-sun Hwang*

ABSTRACT

This paper considers one-sample and two-sample test for the logistic function by means of Kolmogorov-Smirnov type statistics. The standard tables used for the Kolmogorov-Smirnov test are valid only when the function is completely specified; but they are not valid if the parameters of function are estimated from the sample. This note presents modified tables for the Kolmogorov-Smirnov type statistic. These tables can be used to test the hypothesis that a sample comes from a logistic function when shape parameter(α) and location parameter (β) must be estimated from the sample by the method of maximum likelihood. Monte Carlo method is employed to calculate the critical values of the test. The tables of the critical values are provided.

1. Introduction

The logistic growth function

$$P=\frac{1}{1+\beta e^{-\alpha x}}, \quad -\infty<x<\infty, \quad \alpha>0, \quad \beta>0, \tag{1.1}$$

has been extensively studied in certain applications of economics, biology and ecology. Since 1920 fitting a logistic curve has been attempted utilizing such methods as the least-squares, the maximum likelihood, and others under the assumption that the logistic model is correct. However, the validity of the logistic model is not determined. Therefore, it is interesting to investigate whether or not a sample came from a logistic function, or to test whether or not the two independent samples came from the same logistic population. The former is called the one-sample test, and the latter the two-sample test.

* Department of Computer Science, Korea University, Seoul 132, Korea

The logistic one-sample test can be considered as a test of goodness of fit between the sample logistic function and the population logistic function. This test involves using the value of the observed probability of the sample logistic function for each x_i, and determining the point 'x_i' at which the "expected and sample estimated p_i's" show the greatest divergence.

The two-sample test can be conceived of a test of the agreement between two-test data samples. If the two samples have been drawn from the same logistic function, the sample estimates of the probabilities of success p_i for both samples may be expected to be fairly close to each other. If these values are "too far apart" at any point of x, the data suggest that the samples came from different logistic populations. Thus, a large difference between any two different estimates of p_i would constitute an evidence for rejecting the null hypothesis that the two samples came from the same logistic population.

To develop these tests, the two sample test will be considered first. It is known that the "Standard Logistic Function" is defined as

$$p=\frac{1}{1+e^{-y}},\quad -\infty<y<\infty. \tag{1.2}$$

The logistic function can be standardized by taking $\alpha=1$ and $\beta=1$, or by simply taking $y=-\log\beta+\alpha x$ in (1.1). For the two samples, α and β values are estimated from the sample statistics (a_0, b_0) and (a_1, b_1) respectively, by means of the maximum likelihood or equivalent methods and by plugging these estimates into (1.2) the maximum deviation D is then computed by:

$$D=\text{Max}\,|p_0(y)-p_1(y)|,$$

where $p_0(y)$ is the estimate of the probability of success at $x=x_i$ obtained from the first sample, and $p_1(y)$ is obtained from the second sample.

The following rule is set up: If the value of D exceeds the critical value in the table the null hypothesis will be rejected with conclusion that the sample came from different logistic populations.

The test procedures are summarized as follows:

1. Obtain a sample of n_i dichotomous observations at each x_i.
2. Estimate the parameters (α_j and β_j, $j=0, 1$) from the two samples.
3. Evaluate y_{0i} and y_{1i},

where $y_{0i}=-\log b_0+a_0 x_i$ and $y_{1i}=-\log b_1+a_1 x_i$.

4. Determine D, where $D=\text{Max}\,|p_0(y_{0i})-p_1(y_{1i})|$.
5. If the value of D exceeds the critical value in the table, reject the hypothesis.

Critical values of D (see Appendix) are calculated by the sampling distribution of D through Monte Carlo method by using the standard logistic model. The significance of a given value of D depends on n and M (n is the sample size at each x_i, and M is the number of x). In the case of the sample size the value of D that is not listed in the table can be approximated by interpolation. The calculation was performed on the CDC CYBER 70/74.

When the values are compared with those in the standard table for the Kolmogorov-Smirnov test (Birubaum (1952), Massey (1951)) it is found that the ratio of the Monte Carolo values to the standard values remains relatively fixed. It appears that the Monte Carlo critical values are in most cases approximately half the standard values, especially for the value of $M=5$, and for the value of $M=7$, the critical values are approximately two thirds the standard values.

As a next step, the power of the test will be examined by using Monte Carlo method. The probabilitiy of rejecting the hypothesis of logistic model using D Statistic as compared with Chi-Square Statistic when sample size is 30 is given in Table 1. This tabled values reveal that the power of our K-S type test is larger than that of x^2-test if the underlying population is normal. The Monte Carlo calculations are based upon 1000 sample runs for each population.

Table 1. Power of Test

Underlying Population	K-S Type Test		Chi-Square Test	
	$\alpha=.05$	$\alpha=.10$	$\alpha=.05$	$\alpha=.10$
Logistic	.05	.10	.05	.10
Normal	.31	.42	.25	.30

2. One Sample Test

Let $P_0(x)$ be a observed probability and $P_1(x)=\hat{P}(x)$ be the logistic function with $\hat{\alpha}=a$, shape parameter, and $\hat{\beta}=b$, location parameter. The parameters a and b are estimated from the sample by using maximum likelihood estimate through Newton-Raphson iteration or equivalent methods. The value of $P_1(y_i)=P_1(x_i)$ is the estimated probability at x_i, namely,

$$P_1(y_i)=\frac{1}{1+e^{-y_{1i}}}, \quad i=1,2,\ldots,M,$$

where $y_{1i}(x)=-\log b+ax_i$. Under null hypothesis it is assumed that the sample logistic function has been drawn from the specified logistic function. It is expected that $P_1(y)$ should be close to $P_0(x)$ for every value of x. Under H_0 it is expected that the difference between $P_0(x)$ and $P_1(y)$ should be small for all x's. The H_0 is rejected if the difference is too large for some x in the sense of absolute value. This test is based on the largest value of $|P_0(x)-P_1(y)|$, which is called the K-S type maximum absolute deviation, D, where $D=\text{Max}|P_0(x)-P_1(y)|$.

3. Example

3.1 Example 1 (One-Sample Test)

The data to illustrate this test are obtained from a random sample generated from a logistic model for which the parameters are arbitrarily specified as following:

$\alpha=2$ and $\beta=50$;

$x=.5,\ 1.0,\ 1.5,\ 2.0,\ 2.5,\ 3.0,\ 3.5$;

$M=7$(We have seven points of x);

$n=30$ (30 observations at x_1).

A sample function generated by a random sample is shown in Table 2.

Table 2. Sample Data

i	x_i	# of 0	# of 1	$P_0(x_i)$
1	0.5	29	1	.03333
2	1.0	24	6	.20000
3	1.5	20	10	.33333
4	2.0	14	16	.53333
5	2.5	8	22	.73333
6	3.0	2	28	.93333
7	3.5	1	29	.96667

The parameters α and β in this sample function estimated by the direct maximum likelihood method are:

$$a=1.87486 \qquad b=34.38890$$

Now y_{1i}, $P_1(y_{1i})$ values are calculated as follows:

$$y_{1i}=-\log\ 34.3889+1.87486\,X_i$$

$$P_1(y_{1i})=\frac{1}{1+e^{-y_i}},\quad i=1,2,\ldots\ldots,M$$

The largest of these difference, D, is obtained from the last column in Table 3, which is

$$D=\text{Max}\,|P_0(x)-P_1(y)|=0.04371.$$

This value is compared with the tabulated value for the specified level of significance, $\alpha=.05$, i.e., $D_{\alpha=.05(M=7,n=30)}=.175$. Since the tabulated value $D=.175$ is greater than the calculated value of $D=.04371$, the null hypothesis of the logistic model is not rejected at 5%level of significance, and the function is estimated by

$$P=\frac{1}{1+34.3889e^{-1.8748x}}$$

Table 3. Calculation of D statistic

$M=7$
$a=1.87486,\ b=34.38890$

i	x_i	$P_0(x_i)$	y_i	$P_1(y_i)$	$D(i)$
1	.5	.03333	−2.60030	.06912	.03579
2	1.0	.20000	−1.66287	.15938	.04062
3	1.5	.33333	−.72544	.32620	.00713
4	2.0	.53333	.21199	.55230	.01897
5	2.5	.73333	1.14942	.75940	.02607
6	3.0	.93333	2.08685	.88962	.04371
7	3.5	.96667	3.02428	.95366	.01301

3.2 Example 2 (Two-Sample Test)

The precedure of the two-sample test follows the same pattern as that of the one-sample test: but this time it is necessary to estimate the parameters $(\alpha_0,\ \beta_0)$, $(\alpha_1,\ \beta_1)$,

Table 4. Second Sample Data

i	x_i	# of 0	# of 1	Sample $P_{1i}(x_i)$
1	.5	27	3	.10000
2	1.0	24	6	.20000
3	1.5	23	7	.23333
4	2.0	17	13	.43333
5	2.5	15	15	.50000
6	3.0	10	20	.66667
7	3.5	9	21	.70000

Table 5 Calculation of D Statistic

$M=7$
$a_0=1.87486$, $b_0=34.38830$
$a_1=1.01112$, $b_1=12.01860$

i	x_i	y_{0i}	$P_0(y_{0i})$	y_{1i}	$P_1(y_{1i})$	$D(i)$
1	.5	−2.60030	.06812	−1.98090	.12122	.05310
2	1.0	−1.66287	.15938	−1.47534	.18613	.02675
3	1.5	−.72544	.32620	−.96978	.27433	.05187
4	2.0	.21199	.55280	−.46422	.38533	.16747
5	2.5	1.14342	.75940	.04134	.51033	.24907
6	3.0	2.08685	.88962	.54690	.63342	.25620
7	3.5	3.02422	.95366	1.05246	.74125	.21241

independently from two different observed samples. The sample data of Table 2 will be used as the first sample, and the newly generated second set of sample data is given in Table 4.

Using the direct maximum likelihood estimate, the following results are obtained:

First sample......$a_0=1.87486$, $b_0=34.38890$,

Second sample......$a_1=1.01112$, $b_1=12.01860$.

Then the values for y_{0i}, $P_0(y_{0i})$, y_{1i} and $P_1(y_{1i})$ can be computed through appropriate substitution.

Consider the difference, $D(I)$, in the last column of Table 5. The largest of the differences is

$$D=\text{Max}|P_0(y_0)-P_1(y_1)|=.256.$$

Reference to the D tables (Appendix) reveals that the critical value associated with $M=7$, $n=30$ and $\alpha=.05$ is .175. Since the D value is larger than the critical value, the null hypothesis of equal logistic population is rejected: that is, there is a significant difference between these two logistic functions. Thus, it is concluded that the two samples have not come from the same function, and that the test data should not be considered as coming from the same logistic.

4. Appendix: Table of Critical Values of D

The values of D given in the tables are critical values associated with selected values of n, each table using a different value of M. Any value of D which is equal to or

Table of Critical Values of D
($M=5$)

Sample Size of Each y n	Level of Significance for $D=\mathrm{Max}\|P_0(y)-P_1(y)\|$				
	.20	.15	.10	.05	.01
5	.196	.215	.251	.296	.400
8	.176	.190	.220	.252	.340
10	.162	.174	.196	.227	.300
12	.150	.160	.179	.207	.272
15	.134	.143	.160	.186	.245
20	.116	.125	.137	.162	.213
25	.104	.113	.123	.145	.188
30	.092	.101	.111	.132	.170
35	.085	.094	.104	.122	.158
40	.079	.087	.097	.113	.149
45	.075	.082	.091	.105	.140
50	.071	.078	.086	.099	.132
100	.052	.057	.064	.072	.087

Table of Critical Values of D
($M=7$)

Sample Size of Each y n	Level of Significance for $D=\mathrm{Max}\|P_0(y)-P_1(y)\|$				
	.20	.15	.10	.05	.01
5	.283	.303	.326	.359	.420
8	.239	.254	.276	.308	.380
10	.219	.233	.254	.285	.353
12	.202	.216	.236	.267	.328
15	.180	.190	.120	.240	.295
20	.154	.163	.175	.205	.250
25	.143	.152	.163	.190	.233
30	.132	.141	.152	.175	.216
35	.122	.131	.142	.160	.201
40	.112	.121	.132	.147	.186
45	.105	.114	.124	.138	.172
50	.099	.108	.116	.129	.160
100	.072	.076	.084	.092	.111

greater than the tabulated value is significant at the indicated level of significance. These values were obtained as a result of the Monte Carlo calculation, using 2,000 samples. n is the numbers of samples at x_i (n=5, 8, 10, 12, 15, 20, 25, 30, 40, 45, 50, 100). For the sample size that is not shown in the table the value of D can be approximated by interpolation.

Table of Critical Values of D
($M=9$)

Sample Size of Each y n	Level of Significance for $D=\text{Max}\vert P_0(y)-P_1(y)\vert$				
	.20	.15	.10	.05	.01
5	.306	.324	.349	.389	.445
8	.265	.282	.305	.341	.395
10	.240	.256	.276	.306	.360
12	.223	.238	.258	.285	.332
15	.200	.213	.233	.256	.300
20	.173	.183	.197	.217	.265
25	.158	.167	.180	.200	.235
30	.145	.154	.164	.184	.213
35	.134	.143	.153	.172	.197
40	.124	.133	.143	.160	.185
45	.116	.124	.133	.150	.175
50	.110	.115	.124	.140	.165
100	.079	.083	.090	.102	.120

Table of Critical Values of D
($M=11$)

Sample Size of each y n	Level of Significance for $D=\text{Max}\vert P_0(y)-P_1(y)\vert$				
	.20	.15	.10	.05	.01
5	.325	.343	.365	.401	.466
8	.284	.299	.318	.351	.425
10	.260	.274	.293	.325	.393
12	.238	.252	.270	.301	.363
15	.216	.227	.242	.270	.323
20	.190	.201	.215	.240	.280
25	.166	.177	.191	.215	.253
30	.153	.164	.176	.196	.234
35	.140	.151	.162	.180	.216
40	.132	.140	.151	.165	.201
45	.125	.132	.142	.155	.186
50	.119	.126	.135	.148	.174
100	.083	.088	.095	.105	.127

Table of Critical Values of D
$(M=15)$

Sample Size of Each y n	Level of Significance for $D=\text{Max}\mid P_0(y)-P_1(y)\mid$				
	.20	.15	.10	.05	.01
5	.335	.352	.374	.405	.490
8	.289	.305	.326	.356	.435
10	.265	.279	.300	.329	.400
12	.244	.255	.276	.304	.365
15	.218	.229	.249	.276	.335
20	.190	.201	.214	.237	.295
25	.171	.182	.193	.214	.260
30	.154	.164	.175	.195	.239
35	.142	.151	.162	.180	.220
40	.135	.143	.153	.170	.215
45	.128	.135	.145	.160	.193
50	.123	.130	.140	.154	.182
100	.085	.090	.096	.108	.134

REFERENCE

(1) Birnbaum, Z.W. (1952). Numerical Tabulation of the Distribution of Kolmogorov's Statistic for Finite Sample Size, *J. Amer. Statist. Assoc.* Vol. 47, 425-441.

(2) Lillifors, H. (1967). On the Kolmogorov-Smirnov Test for Normality with Mean and Variance Unknown, *J. Amer. Statist. Assoc.* Vol. 62, 399-402.

(3) Lillifors, H. (1969). On the Kolmogorov-Smirnov Test for the Exponential Distribution with Mean Unknown, *J. Amer. Statist. Assoc.* Vol. 64, 387-389.

(4) Massey, F.J. (1951). The Kolmogorov Test for Goodness of Fit, *J. Amer. Statist. Assoc.* 68-78.

(5) Smirnov, N. (1948). Table for Estimating the Goodness of Fit of Empirical Distribution. *Annals of Math. Stat.*, Vol. 19, 279-281.