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Multi-Dimensional Local Limit Theorems for Large Deviations[+]

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ABSTRACT

In analogy to the theorem proved by So and Jeon (1982), we give a multi-dimensional version of local limit theorem for large deviations in the continuous case. We also prove a similar theorem in the case of lattice random vectors. Some examples are given.

1. Introduction

In one dimensional case, So and Jeon(1982) proved a local limit theorem for large deviations for i.i.d. continuous random variables which has several merits over the result by Richter (1957). In this paper, we extend this to the multi-dimensional case and obtain similar results both for the continuous case and for the lattice case, keeping the same merits over the multivariate results by Richter (1958).

2. Main Results

In the multi-dimensional case, we have the following local limit theorem for large deviations which is analogous to that proved in the paper by So and Jeon (1982).

The following notations shall be used.

$\underset{\sim}{x}=(x_i)$ is m-dimensional (column) vector, $|\underset{\sim}{x}|=\sup_{i\leq j\leq m}|x_j|$ is the sup-norm of vector $\underset{\sim}{x}$, $d\underset{\sim}{x}=dx_1\ldots dx_m$, $\langle \underset{\sim}{s}, \underset{\sim}{t}\rangle=\sum_{i=1}^{m} s_i t_i$ is the scalar product of the vectors $\underset{\sim}{s}$ and $\underset{\sim}{t}$.

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Theorem 1. Let $X^{(1)}$, $X^{(2)}, \dots$ be i.i.d. m-dimensional random vectors with a common distribution function F. Let $\underset{\sim}{c}$ be a m-dimensional vector. Let the following conditions hold:

1) $\phi(\underset{\sim}{s})=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\langle\underset{\sim}{s},\underset{\sim}{x}\rangle\, dF(\underset{\sim}{x})<\infty$ for $\underset{\sim}{s}\in N$,

where N is an open convex set containing zero.

2) There is a $\underset{\sim}{\tau}\in N$ such that

$$\text{grad}\log\phi(\underset{\sim}{\tau})=\underset{\sim}{c}.$$

3) There is a positive integer n_0 such that

$$\sup_{\underset{\sim}{s}\in N}\int_{-\infty}^{\infty}\left|\frac{\phi(\underset{\sim}{s}+i\underset{\sim}{t})}{\phi(\underset{\sim}{s})}\right|^{n_0}dt<\infty.$$

Let $f_n(\cdot)$ be the p.d.f. of $\dfrac{X^{(1)}+\dots+X^{(n)}}{n}$ for $n\geq n_0$ which exists by 3).

Then, for $\underset{\sim}{w}=\underset{\sim}{w}_n=o(1)$, we have, as $n\to\infty$,

$$f_n(\underset{\sim}{c}+\underset{\sim}{w})=\frac{n^{m/2}}{(2\pi)^{m/2}\Delta^{1/2}}\exp\{-nr(\underset{\sim}{c}+\underset{\sim}{w})\}\{1+o(1)\},$$

where $r(\underset{\sim}{a})=\sup_{\underset{\sim}{s}\in N}\{\langle\underset{\sim}{s},\underset{\sim}{a}\rangle-\log\phi(\underset{\sim}{s})\}$

and $\Delta=\det||\partial^2\log\phi(\underset{\sim}{\tau})/\partial\tau_i\,\partial\tau_j||>0.$

Remark. The condition 3) above seems to be somewhat stronger than the condition(A) of Richter (1958) since the former implies the latter clearly. In many examples in which Richter's condition (A) is satisfied, however, condition 3) is also obtained.

In the case of random vectors on a lattice, we have an analogous local limit theorem.

Theorem 2. Let $X^{(1)}$, $X^{(2)}, \dots$ be i.i.d. m-dimensional random vectors which take only integral values $\underset{\sim}{k}$ (vectors with integral coordinates). Let $\underset{\sim}{c}$ be a m-dimensional vector and $p(\underset{\sim}{k})=p\{X^{(j)}=\underset{\sim}{k}\}$. Let the following conditions hold:

1) The distribution of $X^{(j)}$ is unilatticed, that is, the greatest common divisor of the $m!$ times the volumns of the m-dimensional simplices, all $m+1$ vertices of which lie in integral points of $\underset{\sim}{k}$ for which $p(\underset{\sim}{k})>0$, is equal to 1.

2) $\phi(\underset{\sim}{s})=\sum_{\underset{\sim}{k}}p(\underset{\sim}{k})\exp\langle\underset{\sim}{s},\underset{\sim}{k}\rangle<\infty$ for $\underset{\sim}{s}\in N$,

where N is an open convex set containing zero.

3) There is a $\underset{\sim}{\tau}\in N$ such that

$$\text{grad}\log\phi(\underset{\sim}{\tau})=\underset{\sim}{c}.$$

Let $p_n(\underset{\sim}{k})=p\{\sum_{i=1}^{n}X^{(i)}=\underset{\sim}{k}\}$ and

$$\underset{\sim}{x}=\underset{\sim}{x}_{n,k}=\underset{\sim}{k}/n=\underset{\sim}{c}+\underset{\sim}{w}.$$

Then, for $\underset{\sim}{w}=o(1)$, we have, as $n\to\infty$,

$$p_n(\underset{\sim}{k})=\frac{1}{(n)^{m/2}(2\pi)^{m/2}\Delta^{1/2}}\exp\{-nr(\underset{\sim}{c}+\underset{\sim}{w})\}\{1+o(1)\},$$

where $r(\underset{\sim}{a})=\sup_{\underset{\sim}{s}\in N}\{\langle\underset{\sim}{a},\ \underset{\sim}{s}\rangle-\log\phi(\underset{\sim}{s})\}$

and $\Delta=\det\|\partial^2\log\phi(\underset{\sim}{\tau})/\partial\tau_i\,\partial\tau_j\|>0.$

Remark. Condition 1) and 2) above are the same as those of Theorem 2 of Richter (1958).

3. Proofs

The proof of Theorem 1 is essentially same as that of So and Jeon (1982) and is therefore omitted. To prove Theorem 2, we use the multivariate version of Lemma 1 in So and Jeon(1982). Then, for all $\underset{\sim}{s}\in N$,

$$\left\{\frac{\phi(\underset{\sim}{s}+i\underset{\sim}{t})}{\phi(\underset{\sim}{s})}\right\}^n=\sum_{\underset{\sim}{k}}\frac{e^{\langle\underset{\sim}{s},\,\underset{\sim}{k}\rangle}}{\phi(\underset{\sim}{s})^n}p_n(\underset{\sim}{k})\ \exp\ i\langle\underset{\sim}{t},\ \underset{\sim}{k}\rangle.$$

Consequently

$$(1)\qquad p_n(\underset{\sim}{k})=\frac{\exp\langle\underset{\sim}{s},\ \underset{\sim}{k}\rangle}{(2\pi)^m}\phi^{-n}(\underset{\sim}{s})\int_{|\underset{\sim}{t}|<\pi}\left\{\frac{\phi(\underset{\sim}{s}+i\underset{\sim}{t})^n}{\phi(\underset{\sim}{s})}\right\}\exp(-i<\underset{\sim}{t},\ \underset{\sim}{k}>)dt$$

for all $\underset{\sim}{s}\in N$.

Choose, for each n, $\tau_n\in N$ such that

$$\text{grad}\ \log\ \phi(\underset{\sim}{\tau}_n)=\underset{\sim}{x}_n=\underset{\sim}{c}+\underset{\sim}{w}_n.$$

And decompose the last term of (1) into two parts as follows:

$$I_n=\int_{|\underset{\sim}{t}|\leq\pi}\left\{\frac{\phi(\underset{\sim}{s}+i\underset{\sim}{t})}{\phi(\underset{\sim}{s})}\right\}^n\exp(-i<\underset{\sim}{t},\ \underset{\sim}{k}>)d\underset{\sim}{t}$$

$$=\int_{|\underset{\sim}{t}|\leq\varepsilon}\left\{\frac{\phi(\underset{\sim}{s}+i\underset{\sim}{t})^n}{\phi(\underset{\sim}{s})}\right\}^n\exp(-i\langle\underset{\sim}{t},\ \underset{\sim}{k}\rangle d\underset{\sim}{t}+R_n,$$

where $$R_n=\int_{\varepsilon<|\underset{\sim}{t}|\leq\pi}\left\{\frac{\phi(\underset{\sim}{s}+i\underset{\sim}{t})^n}{\phi(\underset{\sim}{s})}\right\}\exp(-i\langle\underset{\sim}{t},\ \underset{\sim}{k}\rangle)d\underset{\sim}{t}.$$

By condition 1), for every $\varepsilon>0$, there exists a $\alpha(\varepsilon)>0$ such that for $\varepsilon<|t|\leq\pi$,

$$\left|\frac{\phi(\underset{\sim}{s}+i\underset{\sim}{t})}{\phi(\underset{\sim}{s})}\right|\leq e^{-\alpha}\ \text{for all}\ \underset{\sim}{s}\ \text{in some neighborhood of}\ \underset{\sim}{\tau}.$$

Therefore

$$|R_n|\leq\int_{\varepsilon<|t|\leq\pi}\left|\frac{\phi(\underset{\sim}{s}+i\underset{\sim}{t})}{\phi(\underset{\sim}{s})}\right|^n d\underset{\sim}{t}$$

(2) $\leq Ke^{-\alpha n} = o(n^{-r})$ for all $r > 0$.

Exactly the same procedure as in So and Jeon(1982) gives us the same estimate of the principal part of I_n. i.e.,

(3) $$I_n \sim \frac{1}{(n)^{m/2}} \int_{R^m} e^{-\langle \underset{\sim}{u}, A\underset{\sim}{u} \rangle} d\underset{\sim}{u}$$
$$\sim \frac{(2\pi)^{m/2}}{(n)^{m/2} \Delta^{1/2}}$$

Substituting (2), (3) into (1) and taking into account that

$$r(\underset{\sim}{x}) = \max_{s \in N} \{\langle \underset{\sim}{x}, \underset{\sim}{s} \rangle - \log \phi(\underset{\sim}{s})\}$$
$$= \langle \underset{\sim}{x}, \underset{\sim}{\tau} \rangle - \log \phi(\underset{\sim}{\tau}),$$

where $\underset{\sim}{\tau}$ is the solution of the saddle point equation

$$\text{grad} \log \phi(\underset{\sim}{\tau}) = \underset{\sim}{x},$$

one has, as $n \to \infty$,

$$p_n(\underset{\sim}{k}) = \frac{1}{(n)^{m/2}(2\pi)^{m/2} \Delta^{1/2}} \exp\{-nr(\underset{\sim}{x})\} \{1 + o(1)\}.$$

Theorem is proved.

4. Examples

In this section, we illustrate Theorems 1 and 2 by two simple examples.

Example 1. Let $X^{(1)}, X^{(2)}, \ldots$ be i.i.d. with common distribution $N_2(\underset{\sim}{\mu}, \Sigma)$, where Σ is nonsingular. Then, we obtain

$$\log \phi(\underset{\sim}{s}) = \langle \underset{\sim}{\mu}, \underset{\sim}{s} \rangle + \langle \underset{\sim}{s}, \Sigma \underset{\sim}{s} \rangle / 2.$$

In this case, the conditions of Theorem 1 are satisfied clearly. Thus by solving saddle point equation

$$\text{grad} \log \phi(\underset{\sim}{s}) = \underset{\sim}{\mu} + \Sigma \underset{\sim}{s} = \underset{\sim}{x},$$

we have

$$r(\underset{\sim}{x}) = \max_{\underset{\sim}{s}} \{\langle \underset{\sim}{x}, \underset{\sim}{s} \rangle - \log \phi(\underset{\sim}{s})\}$$
$$= (\underset{\sim}{x} - \underset{\sim}{\mu}) \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}) / 2.$$

Hence, for each $n \geq 1$,

$$f_n(\underset{\sim}{x}) = n^2 e^{-n(\underset{\sim}{x}-\underset{\sim}{\mu})\Sigma^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu})/2} / 2\pi |n\Sigma|^{1/2} = ne^{-nr(\underset{\sim}{x})} / 2\pi |\Sigma|^{1/2}.$$

Example 2. Let $X^{(1)}, X^{(2)}, \ldots$ be i.i.d. with common tinomial distribution $M\left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Then

$$\phi(s_1, s_2) = \frac{1}{3}(1+e^{s_1}+e^{s_2}).$$

In this case conditions of Theorem 2 are obviously satisfied. Thus by solving saddle point equation

$$\text{grad } \log \phi(\underset{\sim}{s}) = \begin{bmatrix} e^{s_1}/(1+e^{s_1}+e^{s_2}) \\ e^{s_2}/(1+e^{s_1}+e^{s_2}) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we have

$$\begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} = \begin{bmatrix} \log x_1/(1-x_1-x_2) \\ \log x_2/(1-x_1-x_2) \end{bmatrix}.$$

Hence

$$\begin{aligned} r(\underset{\sim}{x}) &= \max_{\underset{\sim}{s}} \{\langle \underset{\sim}{x}, \underset{\sim}{s} \rangle - \log \phi(\underset{\sim}{s})\} \\ &= \langle \underset{\sim}{x}, \underset{\sim}{\hat{s}} \rangle - \log \phi(\hat{s}) \\ &= x_1 \log x_2 + x_2 \log x_2 + (1-x_1-x_2)\log(1-x_1-x_2) + \log 3. \end{aligned}$$

Since $\sum_{i=1}^{n} X^{(i)} \sim M\left(n, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, we obtain

$$np_n(\underset{\sim}{k}) = n(k_1, k_2^n, 1-k_1-k_2)\left(\frac{1}{3}\right)^n.$$

Using Stirling's approximation for factorials, we have, as $n \to \infty$,

$$np_n(\underset{\sim}{k}) = \frac{e^{-nr}(\underset{\sim}{x})}{2\pi\{x_1x_2(1-x_1-x_2)\}^{1/2}} \{1+o(1)\}.$$

Since $\underset{\sim}{x} = \underset{\sim}{x}_{n,k} = \underset{\sim}{c} + o(1)$ and $\Delta^{1/2} = \sqrt{x_1x_2(1-x_1-x_2)}$, we obtain, as $n \to \infty$,

$$np_n(\underset{\sim}{k}) = \frac{e^{-nr}(\underset{\sim}{x})}{2\pi\Delta^{1/2}} \{1+o(1)\}.$$

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