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Asymptotic Expressions for One Dimensional Model of Hemodiafiltration

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=**Abstract**=

The asymptotic solution using the Taylor series has been given explicit form for the solute concentration and overall solute removal in hemodiafilter using one dimensional model. The numerical solutions have been calculated within 0.001% error by the Romberg integration method. Compared with the numerical solutions, the one-term asymptotic solutions were found to be within 3% error for the condition $\frac{\beta}{\alpha} > 3.0$ and three-terms asymtotic solutions were required for the condition $\frac{\beta}{\alpha} > 0.7$ where β denotes measure of convection over diffusional transport and α the ratio of blood flow rate over dialysate flow rate.

1. Introduction

In the past twenty years several models have been proposed for solute removal in a parallel plate dialyzer. These models fall into two classes; one is one-dimensional model where average axial velocity is used[1,2] and the other is two-dimensional model which uses the poiseulle type velocity profile [3]. Recently it has become possible to perform diafiltration in place of dialysis when it is desired to remove toxic middle molecules from artificial kidney patients. This technique, sometimes called as crossflow filtration, uses ultrafiltration in addition to diffusion for the removal of solutes.

Popovich et al. [4] carried out a numerical analysis for the two-dimensional model of diafiltration with the constant flux of ultrafiltration along the axis. Considering the fact that the channel length is much larger than the channel height in the dialyzer, Babb and Scriber[5] proposed one dimensional model for diafiltration, which can be more easily managed than the two-dimensional ones. Jaffrin et al.[6] showed by experiment that one-dimensional model is sufficient for the rate of dialysate larger than 300ml/min. However, still this model is given in an implicit form which can not be utilized readily. In this study first, we derived one-dimensional model using the expression for convection coupled diffusion used by Bean[8] and Klein et al[7]. rather than the mass transfer coefficient and transmittance used previously. And second, the results are given in an explicit form by integrating the terms expanded by the Taylor series and co-

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mpared with those obtained by the Romberg integretion.

2. Asymptotic Expression

(1) Countercurrent system

The schematic flow diagram Figure 1. helps understand the one-dimensional model of diafilter with constant ultrafiltration velocity. Since the ultrafiltation rate along the axis is constant, the fluid flow rates are

$$Q_B = Q_{Bi} - \mathrm{Aux} \qquad (1)$$

$$Q_D = Q_{Di} - \mathrm{Au}(L - x) = O_{Do} - \mathrm{Aux} \qquad (2)$$

The variations in concentrations C_B, C_D are related to the mass transfer across the mee mbrane

$$\frac{d}{dx}(Q_B C_B) = -AJs \qquad (3)$$

$$\frac{d}{dx}(Q_D C_D) = -AJs \qquad (4)$$

$$Js = \frac{u(C_B \exp(u/u_d) - C_D)}{\exp(u/u_d) - 1} \qquad (5)$$

Where u_d is (solute diffusivity/membrane thickness) defined by Bean[8] and Klein et al.[7]. The real value of solute diffusivity can be obtained easily elsewhere[9]. From the above equations, we obtained an ordinary differential equation for the concentration of the blood phase as follows.

$$\frac{dC_B}{dx} + \mathrm{Au}\frac{\frac{1-\sigma\exp(u/u_d)}{Q_B} - \frac{1-\sigma}{Q_D}}{\exp(u/u_d) - 1} C_B = \frac{\mathrm{Au}(1-\sigma)(Q_{Di}C_{Di} - Q_{Bo}C_{Bo})}{Q_B Q_D(\exp(u/u_d) - 1)} \qquad (6)$$

Introducing dimensionless variables as folows

$$\alpha = \frac{Q_{Bi}}{Q_{Di}}, \qquad \beta = \frac{u}{u_d}, \qquad \gamma = \frac{\mathrm{AuL}}{Q_{Bi}}$$

$$C_B^* = \frac{C_B}{C_{Bi}}, \qquad C_D^* = \frac{C_D}{C_{Bi}}, \qquad x^* = \frac{X}{L}$$

$$z^* = \frac{Q_{Di}C_{Di} - Q_{Bo}C_{Bo}}{Q_{Bi}C_{Bi}}$$

we obtain an analytical solution of $C_B^*(x^*)$ from Eq. 6

$$C_B^*(x^*) = \left(1 - \frac{\alpha\gamma x^*}{1+\alpha\gamma}\right)^{-B_1'} (1-\gamma x^*)^{B_2'} \cdot \left\{1 + \int_0^{x^*} \frac{\alpha\gamma(1-\sigma) z^*}{(e^\beta - 1)(1-\gamma x^*)^{B_2'+1}\left(\frac{\alpha\gamma x^*}{1-1+\alpha\gamma}\right)^{1-B_1'}(1+\alpha\gamma)} dx^*\right\} \qquad (7)$$

where

$$B_1' = \frac{1-\sigma}{e^\beta - 1}, \qquad B_2' = \frac{1-\sigma e^\beta}{e^\beta - 1}$$

Since z^* is related to C_{Bo}^*, we can not use *Eq.* 7 directly. From the definition of z^*, C_{Bo}^* becomes

$$C_{Bo}^* = \frac{(1+\alpha\gamma)^{B_1'}(1-\gamma)^{B_2'}\left\{1 + \frac{1}{\alpha} C_{Di}^* \int_0^1 \frac{1}{H(x^*)} dx^*\right\}}{1 + (1+\alpha\gamma)^{B_1'}(1-\gamma)^{B_2'+1}\int_0^1 \frac{1}{H(x^*)}\, dx^*} \qquad (8)$$

where

$$H(x^*) = \frac{(e^\beta - 1)\ (1+\alpha\gamma)\ (1-\gamma x^*)^{B_2'+1}\ (1 - \frac{\alpha\gamma}{1+\alpha\gamma} x^*)^{1-B_1'}}{\alpha\gamma(1-\sigma)}$$

The condition $\gamma \cong 1$ represents that the total cumulated ultrafiltration rate across the membrane becomes nearly the inlet feed flow rate. On the other hand, the condition $\gamma \cong 0$ represents that the total flux across the membrane is negligible. By expanding $\frac{1}{H(x^*)}$ with the Taylor series for $\gamma \cong 0$,

$$\frac{1}{H(x^*)} = M(\gamma x^*) = M(0) + \gamma x^* M'(0) + \frac{(\gamma x^*)^2}{2!} M''(0) + \frac{(\gamma x^*)^3}{3!} M'''(0) + \cdots\cdots$$

$$C_{Bo}^* = \frac{(1+\alpha\gamma)^{B_1'}(1-\gamma)^{B_2'}\left\{1+\frac{1}{\alpha}C_{Di}^*\left[M(0)+\Sigma^i \frac{\gamma^i}{(i+1)!}M^i(0)\right]\right\}}{1+(1+\alpha\gamma)^{B_1'}(1-\gamma)\left[M(0)+\Sigma^i \frac{\gamma^i}{(i+1)!}M^i(0)\right]} \quad (9)$$

The dimensionless reduction of the solute, R becomes

$$R = \frac{Q_{Bi}C_{Bi} - Q_{Bo}C_{Bo}}{Q_{Bi}C_{Bi}} = 1-(1-\gamma)C_{Bo}^*$$

$$= 1 - \frac{(1+\alpha\gamma)^{B_1'}(1-\gamma)^{B_2'+1}\left\{1+\frac{1}{\alpha}C_{Di}^*\left[M(0)+\sum_i \frac{\gamma^i}{(i+1)}M^i(0)\right]\right\}}{1+(1+\alpha\gamma)^{B_1'}(1-\gamma)\ \ M(0(+\Sigma^i \frac{\gamma^i}{(i+1)!}M^i(0)} \quad (10)$$

where $M(0) = \frac{\alpha\gamma(1-\sigma)}{(e^\beta - 1)(1+\alpha\gamma)}$ $\qquad M'(0) = M(0)(B'_2+1) + M(0)(1-B'_1)\frac{\alpha}{1+\alpha\gamma}$

$$M''(0) = M(0)(B'_2+1)(B'_2+2) + 2M(0)\ \ (B_2'+1)\ (1-B_1')\frac{\alpha}{1+\alpha\gamma}$$
$$+ M(0)(1-B_1')(2-B_1')\left(\frac{\alpha}{1+\alpha\gamma}\right)^2$$

$$M'''(0) = M(0)(B_2'+1)(B_2'\ +2)(B_2'+3) + 3M(0)(B_2'+1)(B_2'+2)(1-B_1')\ \frac{\alpha}{1+\alpha\gamma}$$
$$+3M(0)(B_2'+1)\ (2-B_1')\ (1-B_1')\left(\frac{\alpha}{1+\alpha\gamma}\right)^2$$
$$+M(0)(1-B_1')(2-B_1')(3-B_1')\left(\frac{\alpha}{1+\alpha\gamma}\right)^3$$

for $\gamma \cong 1$,

$$\frac{1}{H(x^*} = N(1-\gamma x^*) = NN(0)(1-\gamma x^*)^{-B_2'-1} + NN'(0)(1-\gamma x^*)^{-B_2'}$$
$$+NN''(0)\frac{(1-\gamma x^*)^{-B_2'+1}}{2\,!} + \cdots$$

$$C_{Bo}^* = \frac{(1+\alpha\gamma)^{B_1'}(1-\gamma)^{B_2'}\left\{1+\frac{1}{\alpha}\ C_{Di}^*\left[\frac{NN(O)}{\gamma B_2'}((1-\gamma)^{-B_2'}-1)\quad \sum_i \frac{NN^i(O)}{i!\gamma(B_2'-i)}[(1-\gamma)^{i-B_2}-1)\right]\right.}{1+(1+\alpha\gamma)^{B_1'}(1-\gamma)^{B_2'+1}\left[\frac{NN(O)}{\gamma B'_2}((1-\gamma)^{-B'_2}-1)+\sum_i \frac{NN^i(O)}{i!\gamma(B_2'-i)}((1-\gamma)^{i-B'_2}-1))\right]} \quad (11)$$

$$R = 1 - \frac{(1+\alpha\gamma)^{B_1'}(1-\gamma)^{B_2'+1}\left\{1+\frac{1}{\alpha}\ C_{Di}^*\left[\frac{NN(O)}{\gamma B_2'}((1-\gamma)^{-B_2'}-1)+\sum_i \frac{NN^i(O)}{i!\gamma(B_2'-i)}((1-\gamma)^{i-B_2'}-1)\right]\right.}{1+(1+\alpha\gamma)^{B_1'}(1-\gamma)^{B_2'+1}\left[\frac{NN(O)}{\gamma B_2'}((1-\gamma)^{-B_2'}-1)+\Sigma \frac{NN^i(O)}{i!\gamma\,(B_2'-i)}((1-\gamma)^{i-B_2'}-1)\right]} \quad (12)$$

where $NN(O) = \left(\frac{1+\alpha\gamma}{1+\alpha\gamma-\alpha}\right)^{1-B_1'} \frac{\alpha\gamma(1-\sigma)}{(e^\beta-1)(1+\alpha\gamma)}$

$$NN'(O) = NN(O)\ \frac{\alpha}{1+\alpha\gamma-\alpha}(B_1'-1)$$

$$NN^i(O) = NN(O)\ \left(\frac{\alpha}{1+\alpha\gamma-\alpha}\right)^i \prod_{k=1}^{i} (B_i'-k)$$

(2) **Cocurrent system**

For the cocurrent system, the governing equations are slightly different from those for the countercurrent system,

$$Q_D = Q_{Di} + Aux \quad (13)$$

$$\frac{d}{dx}(Q_D C_D) = AJ_s \quad (14)$$

and

$$z' = Q_{Bi}C_{Bi} + Q_{Di}C_{Di} \quad (15)$$

$$z'^* = \frac{Q_{Bi}C_{Bi} + Q_{Di}C_{Di}}{Q_{Bi}C_B} = 1 + \frac{1}{\alpha}C_{Di}^* \quad (16)$$

We obtain the same form as equation (6),

$$\frac{dC_B^*}{dx^*}+\left(\frac{B_2}{1-\gamma x^*}+\frac{B_1}{1+\alpha\gamma x^*}\right)C_B^* = \frac{B_3}{(1-\gamma x^*)(1+\alpha\gamma x^*)} \qquad (17)$$

where

$$B_1=\frac{\alpha\gamma(1-\sigma)}{e^\beta-1},\quad B_2=\frac{\gamma(1-\sigma e^\beta)}{e^\beta-1},\quad B_3=\frac{\alpha\gamma(1-\sigma)z'^*}{e^\beta-1}$$

From the above ordiuary differential equation, we get the concentration

$$C_B^*=(1-\gamma x^*)^{B_2'}(1+\alpha\gamma x^*)^{-B_1'}\left\{1+\int_0^{x^*}\frac{B_3}{(1+\gamma x^*)^{B_2'+1}(1+\alpha\gamma x^*)^{1-B'_1}}dx^*\right\} \qquad (18)$$

The asymptotic expression of equation (18) is

for $\gamma\cong 0$,

$$C_{B0}^*=(1-\gamma)^{B'_2}(1+\alpha\gamma)^{-B'_1}\left\{1+[F(0)+\sum_i\frac{F^i(0)}{(i+1)!}\gamma^i]\right\} \qquad (16)$$

$$R=1-(1-\gamma)^{B_2'+1}(1+\alpha\gamma)^{-B_1'}\left\{1+[F(0)+\sum_i\frac{F^i(0)}{(i+1)!}\gamma^i]\right\} \qquad (20)$$

where $F(0)=\frac{\alpha\gamma(1-\sigma)z'^*}{e^\beta-1}$

$$F'(0)=F(0)(B_2'+1)+F(0)\alpha(B_1'-1)$$

$$F''(0)=F(0)(B_2'+1)(B_2'+2)+2\alpha F(0)(B_1'-1)(B_2'+1)+\alpha^2F(0)(B_1'-1)(B_1'-2)$$

for $\gamma\cong 1$,

$$C_{B0}^*=(1-\gamma)^{B_2'}(1+\alpha\gamma)^{-B'_1}\left\{1+[\frac{B_3FF(0)}{B_2'\gamma}((1-\gamma)-B_2'-1)+\sum_i\frac{B_3FF^i(0)}{i!(B_2'-i)\gamma}((1-\gamma)^{-B'_2+i}-1)]\right\} \qquad (21)$$

$$R=1-(1-\gamma)^{B'_2+1}(1+\alpha\gamma)^{-B'_1}\left\{1+[\frac{B_3FF(0)}{B'_2\gamma}((1-\gamma)^{-B'_2}-1)+\sum_i\frac{B_3FF^i(0)}{i!(B'_2-i)}((1-\gamma)^{-B'_2+i}-1)]\right\} \qquad (22)$$

where, $FF(0)=(1+\alpha)^{B'_1-1}$

$$FF'(0)=\alpha(1+\alpha)^{B'-2}(1-B'_1)$$

$$FF''(0)=\alpha^2(1+\alpha)^{B'_1-3}(1-B'_1)(2-B'_1)$$

3. Results

In order to get the simple form of the solution, we take only the first term in each bracket [ ] of equations (9), (10), (11), (12), (19), (21) and (22). Figure 2 shows, for $C_{Di}^*=0$, $\sigma=0, \alpha=0.4$ and $\beta=1$, the deviations of the asymptotic solutions from the exact concentration, which was calculated numerically by the Romberg integration method within the relative error of $10^{-3}\%$. This shows that in the countercurrent system, the asymptotic solution of $\gamma\cong 1$ is in good agreement with the exact concentration over all γvalue with maximum error of 2 % which occurs at $\gamma=0.8$, but the asymptotic solution of $\gamma\cong 0$ deviates gradually as γ increases. In the countercurrent system, the asymptotic solution of $\gamma\cong 1$ for C_{B0}^* is well consistent with the numerical one over all γ values with the maximum error of 0.8% which occurs at $\gamma=0.5$ and that of $\gamma\cong 0$ is also consistent with the maximum error of 3.34% at $\gamma=0.8$. And in view of the removal efficiency, the countercurrent system is superior to the cocurrent system. Generally, for the artificial kidney system, the inlet concentration of the solute in the dia lysate stream is nearly zero and reflection coefficient σ varies from zero to 1, α is smaller than 0.5 and β is larger than 0.1. If $\frac{\beta}{\alpha}$is greater than 3.0 the one-term asymptotic solutions of equations(11), (12), (21) and (22) for concentration and reduction are in good agreement with the exact solutions of the cocurrent and countercurrent systems within 3% error. By taking 3 terms in each bracket [] of equations (11), (12), (21) and (22), for $\frac{\beta}{\alpha}>0.7$, we can obtain satisfact-

Table 1. The Exact and Asymptotic Values of C_{B0}^* and Reduction

parameter			γ	0.02	0.05	0.1	0.2	0.5	0.8	0.9	0.96	0.98
	counter-	exact	C_{B0}^*	1.0042	1.0108	1.0224	1.0483	1.1600	1.4162	1.6454	2.0022	2.3197
$\sigma=0.1$	current		Red(%)	1.6849	3.9732	7.9851	16.1390	41.9984	71.6767	83.5463	91.9913	95.360
		1 term	C_{B0}^*	1.0042	1.0107	1.0222	1.0479	1.1594	1.4156	1.6450	2.0019	2.3195
$\beta=1.0$		asympto tic(γ=1.	Red(%)	1.5892	3.9835	8.0037	16.1690	42.0312	71.6880	83.5504	91.9923	95.3611
	co-	exact	C_{B0}^*	1.0042	1.0108	1.0224	1.0483	1.1606	1.4216	1˙6587	2.0325	2.3679
	current		Red(%)	1.5849	3.9732	7.9847	16.1372	41.9701	71.5689	83.4216	91.8699	95.2642
		1 term asy mptotic	C_{B0}^*	1.0042	1.0107	1.0222	1.0479	1.1596	1.4198	1.6565	2.0297	2.3646
		(μ=1).	Red(%)	1.5886	3.9823	8.0021	26.1679	42.0199	71.6046	83.4351	91.8812	95.2708
($\beta/\alpha>0.7$)												
	counter-	exact	C_{B0}^*	0.9602	0.9036	0.81 64	0.6627	0.3047	0.06167	0.01663	0.002739	
$\alpha=0.4$	current		Red(%)	5.8983	14.1552	26.5250	46.9846	94.7639	99.8337	99.9890	3term	
$\beta=0.4$		3term asymptotic	C_{B0}^*	00.9602	0.9036	0.8164	0.6627	0.3047	0.06167	0.01663	0.002739	
$\sigma=0.0$		(γ=1.)	Red(%)	5.8993	14.1572	26.5277	46.9870	84.7643	98.7666	99.8337	99.9890	
	co-	exact	C_{B0}^*	0.9602	0.9039	0.8181	0.6738	0.4062	0.3011	0.2893	0.2862	
	current		Red(%)	5.8968	14.1331	26.3736	46.1000	79.6917	93.9777	97.1075	98.8550	
		3term asymptotic	C_{B0}^*	0.9602	0.9039	0.8181	0.6737	0.4062	0.3011	0.2893	0.2862	
		(γ=1.)	Red(%)	5.8971	14.1338	26.3747	46.1015	79.6924	93.9778	97.1075	98.8550	

ory solution within 3% error as shown in Table 1.

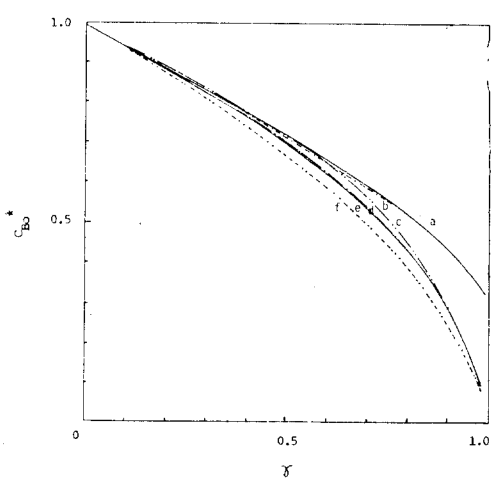

Fig. 1. Mass Transfer in the Cocurent and Countercurrent Systems of Diafilter.

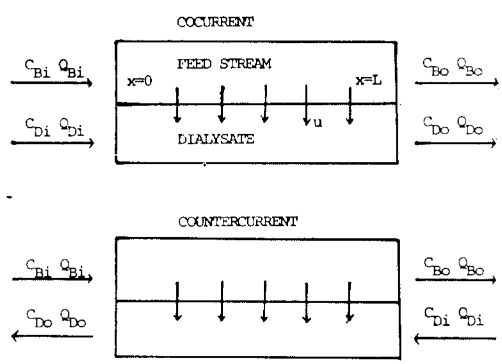

Fig. 2. The Outlet Concentration of Solute versusr. a ; ——Numerical Solution of Cocurrent Flow, b ; ······Asymptotic Solution of Cocurrent Flow($\gamma\cong1.$), c ; —··—Asymptotic Solution of Countercurrent Flow ($\gamma\cong0.$), d ; ——•——Asymptotic Solution of Countercurrent Flow ($\gamma\cong1.$), e ; —— Numerical Solution of Countercurrent Flow, f ; ····· Asymptotic Solution of Cocurrent Flow($\gamma\cong0$)

4. Conclusion

The role of ultrafiltration in solute removal is insignificant for small molecules such as urea but effective for the larger molecules[10]. It is concluded that the outlet concentration and reduction of the middle molecules such as vitamin B_{12} and larger molecules can be calculated analytically within 3% error by using the three-term asymptotic functions of equations (11), (12), (21) and (22).

Nomenclature

A : Membrane area per unit width inthe x direction.

C_B : Solute concentration in the feed stream

C_D : Solute concentration in the dialysate stream.

J_s : Transmembrane solute flux.

L : Axial length of the dialyzer.

Q_B : Flow rate of feed stream at axial position x.

Q_D : Flow rate of dialysate stream at axial position x.

R : Dimensionless reduction of the solute.

u : Ultrafiltration velocity.

u_d : Membrane characteristic (solute diffusivity/membrane thickness)

Subscript Notation

i : inlet

o : outlet

Greek Notation

α(=Alpha) : The ratio of inlet blood flow rate to inlet dialysate flow rate.

β(=Beta) : The ratio of ultrafiltration velocity to u_d

γ(=Gamma) : The ratio of total cumulated ultrafiltratoin rate through the membrane to the inlet blood flow rate.

σ(=Sigma) : Reflection coefficient.

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