604 Transactions of the Korean Society of Mechanical Engineers, Vol. 8, No. 6, pp. 604~609, 1984.

<Technical Paper>

Determination of the Critical Buckling Load of a Circular Ring by the Dynamical Aspect

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(Received March 30, 1984)

動力學的 解析에 의한 圓環의 임계 座屈荷重의 決定

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抄 錄

本 硏究에서는 軸力이 作用하는 圓環의 動力學的 振動 해석으로부터 임계 좌굴 荷重을 유도하였다. 또한 振動 해석에서는 並進 관성 以外에 回轉 관성과 전단 변형 효과를 個別的으로 고려함으로서 두개의 固有 진동수에 관한 公式을 유도하였고 이로부터 좌굴하중을 산출하였다.

그 결과는 Timoshenko 가 재료역학적 이론을 배경으로 산출한 좌굴하중과도 잘 일치함을 확인할 수 있었다.

Nomenclature

A : Area of cross section of a ring
E : Young's modulus
G : Modulus of shear
I : Moment of inertia of cross sectional area
J : Moment of intertia of mass of the ring
K' : Form factor
M : Moment of force
m : Mass of a ring per unit length $m=\rho A/g$
N : Shear force
r : Radial coordinate
S : Axial force due to radial prestress
S_{CR} : Critical buckling force
T : Normal force acting along tangent to a circumference of a ring
t : Time coordinate
u : Radial displacement
w : Tangential displacement

Greek letters

θ : Angular coordinate
Ω : Natural circular frequency of ring vibration
ϕ : Deformation angle due to bending
β : Deformation angle due to shear
ψ : Total deformation angle
ρ : Density of ring material

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1. Introduction

The buckling of the structural elements has long been studied based on strength of materials view point.

Many Scholars contributed in this field of study especially on beams, columns, shells, plates, and circular ring.

During last 20 years, the new approach of the problem was initiated by a group of scholars led by V.V. Bolotin.

This new approach utilizes basically the concept of nonconservative type of force formulation into the dynamical governing equation and treats the problem of buckling as an extreme case of dynamic motion.

However, the most of the non-conservative force[1] analysis of the structural elements were concentrated either on column or beam, we have applied the analysis to the circular ring under axial prestressing force. The classical dynamic analysis on free vibration of circular ring is introduced by Love[2]. Many studies were made to improve and supplement this classical dynamic analysis.

L.L. phillipson[3] considered extensional effect of the ring cross section center line, Kirkhope[4] considered shear deformation effect, K.S. KIM[5] considered rotatory intertia effect, R.R. Archer[6] considered damping effect, C.W. Bert considered rotating ring[7] and S.S. Rao[8] considered rotatory intertia and shear and shear deformation.

However, the effect of prestressing in the free vibration of a circular ring was never considered before.

In this dynamic analysis, we have taken into account of the effects of the translational inertia, the rotatory inertia, and the shear deformation effect separately.

Through the tree separate sets of analysis, we have shown how each effect influence the vibrational frequencies of the circular ring under prestressing condition, and as an extreme condition, we have derived the formula for the buckling load for a circular ring.

One of the practical example of the ring under axial prestress condition is the force fitted ring component.

The result of this study are justified through the comparison with the result of classical analysis by Timoshenko[9] who studied this problem based on strength of materials view point.

While the approach by Timoshenko was basically a statical, he did not need to consider the effects of the inertia terms which we considered together with shear deformation effect in the determination of critical buckling load.

2. Analysis 1

For analysis on the free in-plane flexural vibrations of a thin, elastic and circular ring, we utilize the equations of motion derived by Love[3] and consider only the translational inertia.

In Fig. 1 we have shown an element of ring with symmetrical cross section of unit length with differential angle $d\theta$ and radius R.

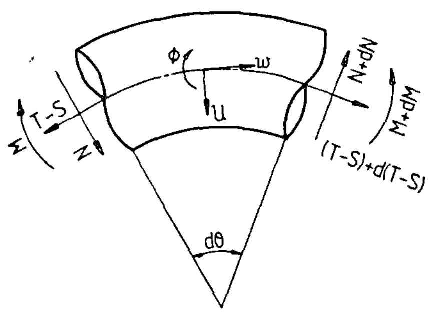

Fig. 1 *Ring element*

From the above mentioned relation, $Rd\theta=1$ holds in Fig. 1.

The radial direction(u direction) equation of motion becomes in the case of compression,

$$-N+\left(N+\frac{\partial N}{\partial\theta}d\theta\right)+\left\{(T-S)+\frac{\partial}{\partial\theta}(T-S)\right.$$

$$d\theta\Big\}d\theta=mR\,d\theta\frac{\partial^2 u}{\partial t^2}$$

Dividing by $R\,d\theta$, we obtain

$$\frac{1}{R}\left\{(T-S)+\frac{\partial N}{\partial\theta}\right\}=m\frac{\partial^2 u}{\partial t^2} \tag{1}$$

where $m=\rho A/g$, the mass per unit length of the ring. On the other hand, the tangential direction(w direction) equation of motion becomes,

$$-(T-S)+\left\{(T-S)+\frac{\partial}{\partial\theta}(T-S)\,d\theta\right\}$$
$$-\left(N+\frac{\partial N}{\partial\theta}\,d\theta\right)d\theta=mR\,d\theta\frac{\partial^2 w}{\partial t^2}$$

Dividing by $R\,d\theta$, we obtain

$$\frac{1}{R}\left\{\frac{\partial}{\partial\theta}(T-S)-N\right\}=m\frac{\partial^2 w}{\partial t^2} \tag{2}$$

The moment equation becomes moment equilibrium equation in this analysis as we neglect rotatnry inertia.

$$-M+\left(M+\frac{\partial M}{\partial\theta}\,d\theta\right)+\left(N+\frac{\partial N}{\partial\theta}\,d\theta\right)Rd\theta=0$$

Dividing by $d\theta$, we obtain

$$\frac{\partial M}{\partial\theta}+NR=0 \tag{3}$$

From the strength of materials theory, moment and deformation have following relation with compressive prestressing force of S[9],

$$-Su+M=\frac{EI}{R^2}\left(\frac{\partial^2 u}{\partial\theta^2}+u\right) \tag{4}$$

Utilizing the geometric relation of deforming ring central cross sectional line[3], the elongation of ring central cross sectional line becomes,

$e=-ud\,\theta+\frac{\partial w}{\partial\theta}d\theta$, using the relation $Rd\theta=1$,

$$d\theta=\frac{1}{R},\quad e=\frac{1}{R}\left(-u+\frac{\partial w}{\partial\theta}\right)$$

Assuming $e=0$, we obtain the inextensional condition as follows.

$$u=\frac{\partial w}{\partial\theta} \tag{5}$$

Substituting M from Eq. (4) into Eq. (3) and solving for N, we obtain

$$N=-\frac{1}{R}\frac{\partial M}{\partial\theta}\left[\left\{\frac{EI}{R^2}\left(\frac{\partial^2 u}{\partial\theta^2}+u\right)\right\}-Su\right] \tag{6}$$

Substituting this N into Eq. (2) and substituting T from Eq. (1), into Eq. (2) and changing u into $\frac{\partial w}{\partial\theta}$ using Eq. (5), the resulting equation of motion becomes,

$$\frac{\partial^6 w}{\partial\theta^6}+2\frac{\partial^4 w}{\partial\theta^4}+\frac{\partial^2 w}{\partial\theta^2}+\frac{R^2S}{EI}\frac{\partial^2}{\partial\theta^2}$$
$$\left(\frac{\partial^2 w}{\partial\theta^2}+w\right)=\frac{mR^4}{EI}\frac{\partial^2}{\partial t^2}\left(w-\frac{\partial^2 w}{\partial\theta^2}\right) \tag{7}$$

By Byerly method, weas sumed solution as,

$$w=w_0e^{i(n\theta+\Omega t)}\quad(n=2,3,4,\cdots) \tag{8}$$

And we obtained following natural frequency relation.

$$n^2\left\{n^4-\left(\frac{R^2S}{EI}+2\right)n^2+\left(\frac{R^2S}{EI}+1\right)\right\}$$
$$=\frac{mR^4}{EI}(1+n^2)\Omega^2 \tag{9}$$

$$\Omega=\left\{\frac{EIn^2(n^2-1)^2}{mR^4(1+n^2)}-\frac{R^2Sn^2(n^2-1)}{mR^4(1+n^2)}\right\}^{1/2} \tag{10}$$

The first term of inside of (9) is the classical solution while the second term shows the effect of prestress. When $\Omega=0$, this represents the buckling condition as shown by Timoshenko and Gere[9].

From equation (1), set

$$\frac{EIn^2(n^2-1)^2}{mR^4(1+n^2)}=\frac{R^2Sn^2(n^2-1)}{mR^4(1+n^2)}$$

$$S_{CR}=\frac{EI(n^2-1)}{R^2} \tag{11}$$

For the lowest mode $n=2$, we obtain the critical value of the compressive force as

$$S_{CR}=\frac{3EI}{R^2} \tag{12}$$

This value is the same as the one obtained by Timoshenko. The equation (10) also shows the so called fluttering natural frequencies of a circular ring under non-conservative type of radial loading S.

2.1. Analysis 2

In this analysis a ring under axial compression with consideration to both translational and rotatory inertia is analysed.

In this way we can see the effect of rotatory inertia more clearly in comparison to the classical love theory. The sets of governing equations are similar from previous analysis except rotatory inertia term in Eq. (14). The governing equations are as follows.

$$\frac{1}{R}\left\{\frac{\theta N}{\partial\theta}+(T-S)\right\}=m\frac{\partial^2 u}{\partial t^2} \tag{13}$$

$$\frac{1}{R}\left\{\frac{\partial(T-S)}{\partial\theta}-N\right\}=m\frac{\partial^2 w}{\partial t^2} \tag{14}$$

$$\frac{\partial M}{\partial\theta}+NR=J\frac{\partial^2\phi}{\partial t^2} \tag{15}$$

$$-Su+M=\frac{EI}{R^2}\left(\frac{\partial^2 u}{\partial\theta^2}+u\right) \tag{16}$$

From the inextensional condition

$$\frac{\partial w}{\partial\theta}=u \tag{17}$$

We obtain the equation of motion under non-conservative follower force S, which is compressive axial force acting at the cross section of ring differential element of unit length.

$$\frac{\partial^6 w}{\partial\theta^6}+2\frac{\partial^4 w}{\partial\theta^4}+\frac{\partial^2 w}{\partial\theta^2}+\frac{R^2 S}{EI}\frac{\partial^2}{\partial\theta^2}\left(w+\frac{\partial^2 w}{\partial\theta^2}\right)$$
$$=\frac{mR^4}{EI}\frac{\partial^2}{\partial t^2}\left(w-\frac{\partial^2 w}{\partial\theta^2}\right)$$
$$+\frac{JR^4}{EI}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^4 w}{\partial\theta^4}+2\frac{\partial^2 w}{\partial\theta^2}+w\right) \tag{18}$$

Assuming

$$w=w_0 e^{i(n\theta+\Omega t)}$$

We obtained following relation.

$$n^2(n^2-1)^2-\frac{R^2 S}{EI}n^2(n^2-1)$$
$$=\frac{mR^4}{EI}(1+n^2)\Omega^2+\frac{JR}{EI}(n^2-1)^2\Omega^2 \tag{19}$$

$$\Omega=\left\{\frac{EIn^2(n^2-1)^2-R^2Sn^2(n^2-1)}{mR^4(1+n^2)+JR(n^2-1)^2}\right\}^{1/2} \tag{20}$$

If we set $\Omega=0$, we obtain critical buckling load Scr as follows.

$$S_{CR}=\frac{EI(n^2-1)}{R^2} \tag{21}$$

For $n=2$, the lowest mode of deformation

$$S_{CR}=\frac{3EI}{R^2} \tag{22}$$

The results obtained from our dynamical analysis conform to the result obtained by Timoshenko. This is physically reasonable, because for the lowest mode of deformation ($n=2$) corresponds only to bending mode as there is no change of sign of $\frac{\partial^2 w}{\partial\theta^2}$.

2.2. Analysis 3

We now analyze the cirtical buckling of the circular ring with only shear deformation in consideration.

The governing equations are

$$\frac{1}{R}\left\{\frac{\partial N}{\partial\theta}+(T-S)\right\}=m\frac{\partial^2 u}{\partial t^2} \tag{23}$$

$$\frac{1}{R}\left\{\frac{\partial}{\partial\theta}(T-S)-N\right\}=m\frac{\partial^2 w}{\partial t^2} \tag{24}$$

$$\frac{\partial M}{\partial\theta}+NR=0 \tag{25}$$

$$-Su+M=\frac{EI}{R}\left(\frac{\partial\phi}{\partial\theta}\right) \tag{26}$$

$$N=K'\beta AG \tag{27}$$

$$\psi=\phi+\beta \tag{28}$$

$$\psi=\frac{1}{R}\left(\frac{\partial u}{\partial\theta}+w\right) \tag{29}$$

$$\frac{\partial w}{\partial\theta}=u \tag{30}$$

Here, S is the radial compressive prestress, N is the shear force acting across section, K' form factor of the cross section, ψ the total deformation, ϕ is the deformation due to bending and β is the deformation due to shear.

From Eq. (27)(28) and (29), we can write ϕ as follows, i,e,

$$\phi=\frac{1}{R}\left(\frac{\partial u}{\partial\theta}+w\right)-\frac{N}{K'AG} \tag{31}$$

Substituting Eq. (31) into Eq. (26) and solving for M, we obtain M as follows

$$M=Su+\frac{EI}{R}\left\{\frac{1}{R}\left(\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial w}{\partial\theta}\right)-\frac{\partial}{\partial\theta}\frac{N}{K'AG}\right\} \quad (32)$$

Substituiing Eq. (32) into Eq. (25) results

$$\frac{\partial}{\partial\theta}\left\{Su+\left(\frac{EI}{R^2}+\frac{EI}{R^2}\frac{\partial w}{\partial\theta}-\frac{EI}{R}\frac{1}{K'AG}\frac{\partial N}{\partial\theta}\right)\right\}+NR=0$$

Dividing by R and rearranging, we get

$$\left(1-\frac{EI}{R^2K'AG}\frac{\partial^2}{\partial\theta^2}\right)N+\frac{S}{R}\frac{\partial u}{\partial\theta}+\frac{EI}{R^3}\frac{\partial^3 u}{\partial\theta^3}+\frac{EI}{R^3}\frac{\partial^2 w}{\partial\theta^2}=0 \quad (33)$$

We have have now combined Eq. (25)(26)(27) (28) and (29) int one Eq. (33).

From Eq. (23)

$$(T-S)=mR\frac{\partial^2 u}{\partial t^2}-\frac{\partial N}{\partial\theta} \quad (34)$$

Substitution Eq. (34) into Eq. (24) yields

$$\frac{1}{R}\left\{\frac{\partial}{\partial\theta}\left(mR\frac{\partial^2 u}{\partial t^2}-\frac{\partial N}{\partial\theta}\right)-N\right\}=m\frac{\partial^2 w}{\partial t^2}$$

Multiplying both sides by R and rearranging, we get

$$-\left(\frac{\partial^2}{\partial\theta^2}+1\right)N-mR\frac{\partial^2 w}{\partial t^2}+mR\frac{\partial}{\partial\theta}\frac{\partial^2 u}{\partial t^2}=0 \quad (35)$$

We have condensed relation of Eq. (23) and (24) into Eq. (35) So, the relation from Eq. (23) upto (29) are combined as Eq. (33) and (35).

We can remove the terms including N by applying the linear operator $\left(\frac{\partial^2}{\partial\theta^2}+1\right)$ to Eq. (33)and another operator $\left(1-\frac{EI}{R^2K'AG}\frac{\partial^2}{\partial\theta^2}\right)$ to Eq. (35) and add two equations.

This operation leaves the following terms.

$$\left(\frac{\partial^2}{\partial\theta^2}+1\right)\left\{\frac{S}{R}\frac{\partial u}{\partial\theta}+\frac{EI}{R^3}\frac{\partial^3 u}{\partial\theta^3}+\frac{EI}{R^3}\frac{\partial^2 w}{\partial\theta^2}\right\}+\left(1-\frac{EI}{R^2K'AG}\frac{\partial^2}{\partial\theta^2}\right)\left\{mR\frac{\partial}{\partial\theta}\frac{\partial^2 u}{\partial t^2}-mR\frac{\partial^2 w}{\partial t^2}\right\}=0$$

Rearranging and expanding these terms, and using inextensional condition Eq. (30), we can obtain one single governing equation.

$$\frac{\partial^6 w}{\partial\theta^6}+2\frac{\partial^4 w}{\partial\theta^4}+\frac{\partial^2 w}{\partial\theta^2}+\frac{R^2S}{EI}\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial^2 w}{\partial\theta^2}+w\right)=\frac{mR^4}{EI}\frac{\partial^2}{\partial t^2}\left(w-\frac{\partial^2 w}{\partial\theta^2}\right)+\frac{mR^2}{K'AG}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^4 w}{\partial\theta^4}-\frac{\partial^2 w}{\partial\theta^2}\right) \quad (36)$$

We assume a solution for this governing equation as

$$w=w_0e^{i(n\theta+\Omega t)}$$

Substituting this solution into Eq. (36) resultes Ω as follows.

$$\Omega=\left\{\frac{EIn^2(n^2-1)^2-R^2Sn^2(n^2-1)}{mR^4(1+n^2)+\{EImR^2n^2(n^2+1)/K'AG\}}\right\}^{1/2} \quad (37)$$

If we set $\Omega=0$, we obtain critical buckling load Scr as follows.

$$S_{CR}=\frac{EI(n^2-1)}{R^2} \quad (38)$$

When we set $n=2$, we obtain critical buckling load for the circular ring.

$$S_{CR}=\frac{3EI}{R^2} \quad (39)$$

3. Conclusion

We have applied the dynamic analysis to the problem of critical buckling load of a circular ring. In three separate analysis, we have obtained corresponding frequency formulas under prestressing force with consideration on translational inertia, and shear effect. As a result, we have shown that these effects do not change the critical buckling load in the extreme while these effects do influence only the vibrational frequencies in the course of reaching the critical buckling load.

The analysis performed in this study confirmed the reliablity of the classical theory on buckling load of the circular ring and derived new formulas on natural frequency considering the same effects mentioned above.

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