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An Analysis of the Fracture Initiation of Falling Type Impact Test for Toughened Rigid Plastics

Jin Woo Kim*

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인성의 강소성 플라스틱 재료에 대한 낙하충격 시험의 파괴개시에 관한 연구

김 진 우

초 록

다트식 낙하충격 시험에 있어서 인성의 강소성 플라스틱 재료의 준정적 선형의 점탄성 모델이 구성되어 해석되었다. 완화계수함수, $E(t)=E_f+(E_0-E_f)e^{-t/t_R}$ 형태의 점탄성 재료의 수정된 Maxwell 요소모델을 근거로 충격속도, 파괴에너지, 임계응력등의 중요변수들의 상대적 종속성이 근사계산으로 평가되었다.

1. Introduction

Of the several methods used for evaluating the impact strength of toughened rigid plastics, the falling weight test as detailed for example in ASTM method D 3029~78[1] is generally recognized as providing the most usefut indication of relative impact strength under service conditions. Briefly, the conventional falling weight impact test seeks to determine the energy level of an impacting dart which can be expected to cause fracture in 50 percent of samples tested. The samples are generally plaques prepared by injection molding held in place on the boundary and centrally impacted. Thus there are two main features of a falling weight test which together serve to distinguish it from other types of impact tests such as notched and unnotched Izod and Charpy tests, or conventional high strain rate tension and torsion tests: i) Fracture is initiated on the as molded surface as a result of impact on the opposite surface. Since the stress state is essentially isotropic in the plane of the surface, fracture should initiate naturally at an inherent flaw rather than at an artificial cut or notch. ii) The test is designed to place the samples under conditions at the critical state, i.e. the most severe conditions at which half the population of the samples survives. Unfortunately, a major drawback of the conventional test procedures is that in order to obtain a single impact strength value that is statistically meaningful, a relative large number of identical samples

* Member, Dept. of Mechanical Engineering, Korea Military Academy

must be tested. If one is willing to abandon the requirement that the test be performed at near critical conditions and instead measures the energy lost in fracturing a sample when an excess of energy is furnished to insure fracture, the amount of testing needed to produce statistically meaningful results can be significantly reduced.

Two different methods of performing such tests have been reported recently. One by Gonzalez and Stowell[2] may be described as an instrumented impact test similar to the instrumented Charpy and Izod tests. A transducer (in this caes a strain gage Dynamic Load Cell) is attached to the falling weight and produces a force-time record of the impact. This record may then be interpreted and analyzed and the energy delivered to the sample obtained from a simple calculation involving the impact velocity and the impulse. The latter is found by integrating the force-time record. An alternative method has been used by Dow Chemical and Detroit Testing Labs as a basis for designing an impact tester called a Dymetron. In this instrument gas pressure is used to impart a calibrated amount of energy to a dart which impacts the sample. The unused energy is calculated (actually a table look-up) from the reading of a pressure transducer located in an enclosed cylinder above a piston which travels with the impacting dart.

Both methods have obvious advantages and disadvantages; in particular the instrumented test furnishes a record which can be analyzed and interpreted to yield considerably more information than just the energy used in fracturing the sample, whereas the Dymetron is quick, simple to operate, and furnishes the fracture energy directly. However, in both cases, and to some extent in the conventional falling weight test, the impact conditions (e. g., impact velocity or mass of the impacting dart) are arbitrary. The results are further clouded by the inherent statistical scatter in all three impact tests due to the extreme sensitivity of the fracture process to surface conditions, inhomogeneities, and the distribution of flaws in the samples. A general analysis of falling weight type impact tests would therefore be a useful guide in attempting to correlate results of the various tests completed under differing mechanical conditions.

In the next section a mechanical model of dart impact type tests on toughened rigid plastics is proposed and analyzed. For the sample plate some approximate computations are then performed to assess the relative importance of various parameters such as the impact velocity, fracture initiation energy and critical stress. To this end the load-deflection behavior of the plate is based on the assumption that the material is viscoelastic. Furthermore, introducing a quasistatic linear viscoelastic constitutive relation for small motions we take up the maximum normal stress criterion of fracture initiation which rests on the basic concept of linear elastic fracture mechanics. Finally we examine the plausibility of these assumptions within the frame work of a recent theory of crack initiation and growth in viscoelastic materials due to Shapery[6]

2. Plate Impact Analyses

In either a conventional falling weight impact test or in a Dymetron test we model the sample as a thin circular plate supported at the boundary somewhere between a clamped condition as in the conventional test, and simple supports as in the Dymetron tester. The plate is centrally impacted at $t=0$ by a "dart" of weight W (i.e. mass $m=W/g$) driven by the external force $q(t)$ (in the falling weight test $q(t)=W$, in the Dymetron it is the force of driving pressure less the weight of the piston dart assembly). We denote by $v(t)$ the velocity of the dart and write v_0 for the impact velocity: $v_0=v|_{t=0}$

To simplify the analysis we make the following assumptions:

i) After contact and prior to fracture, the displacement of the center of the plate and the dart are identical. Thus, we are ignoring penetration of the dart into the plate material.

ii) The contact force between dart and plate, denoted $P(t)$, is non decreasing and continuous from impact to fracture initiation.

iii) The load-deflection behavior of the plate can be characterized by a quasi-static linear viscoelastic constitutive relation of the form

$$P(t)=\int_0^t G(t-t')v(t')dt' \tag{1}$$

where $G(t)$, $t\geqslant 0$ represents the plate relaxation function (assuming a rest history for the plate prior to impact)

and $v(t)=\frac{dw}{dt}$ is the central deflection velocity of the plate.

iv) Fracture is initiated when the stresses in the face opposite the impact reach a critical value. In view of the preceding assumptions, we introduce an alternative criterion that fracture is initiated at $t=t_c$ when the load has reached a critical value

$$P(t_c)=P_c \tag{2}$$

We will examine the plausibility of these assumptions later.

Referring to Fig. 1 we write the equation of motion for the dart as

$$q(t)-P(t)=m\frac{dv}{dt} \tag{3}$$

On comparing this with the constitutive equation of the plate (1) and eliminating the contact force $P(t)$ we obtain the following integro-differential equation for the velocity:

$$\frac{dv}{dt}=\frac{1}{m}\left[q(t)-\int_0^t G(t-t')v(t')dt'\right] \tag{4}$$

By using the Laplace transformation of equation (4) and then inverting, we may write

$$v(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{[mv_0+\bar{q}(s)]e^{st}}{ms+\bar{G}(s)}ds \tag{5}$$

$$P(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{[mv_0+\bar{q}(s)]\bar{G}(s)e^{st}}{ms+\bar{G}(s)}ds \tag{6}$$

With the expectation that $P(t)$ is non decreasing (which is consistent with $\frac{dv}{dt}<0$, $q(t)$ essentially constant from impact to fracture) we may calculate the time to initiate fracture implicitely from equation (2). It then follows that the energy imparted to the plate from the initial impact to the initiation of fracture is given by

$$U_c=\int_0^{t_c}P(t)v(t)dt=\int_0^{t_c}\left[q(t)-m\frac{dv}{dt}\right]v(t)dt$$

$$=\int_0^{t_c}q(t)v(t)dt-\frac{1}{2}m\left[v(t_c)^2-v_0^2\right] \tag{7}$$

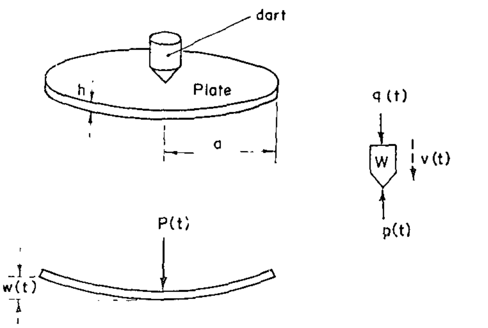

Fig. 1 Free body diagram of dart

3. Some Approximate Computations

In order to study the effect of the initial impact velocity and other parameters on the energy to initiate fracture, an explicit expression for the plate relaxation $G(t)$ is needed. The following approximate analysis should suffice to indicate trends.

For a linearly elastic plate of radius a and thickness h subjected to a concentrated load P at its center, the center deflection is given by Timoshenko[3] as

$$w=\frac{Pa^2k}{16\pi D} \tag{8}$$

where $D=Eh^3/12(1-\nu^2)$ with E the modulus of elasticity and ν Poisson's ratio, and k is a constant reflecting the nature of the boundary supports with $k=1$ for clamped edges and $k=(3+\nu)/(1+\nu)$ for simple supports. Inverting equation (8) we can write

$$P=\frac{4\pi Eh^3w}{3a^2k(1-\nu^2)} \tag{9}$$

For the actual plate material we write $E(t)$ for the uniaxial relaxation modulus and treat Piosson's ratio constant. Then assume that the concentrated force needed to effect a suddenly applied constant central deflection w is approximated by (9) with the elsticity modulus E replaced by the relaxation modulus $E(t)$. Thus

$$G(t)=\frac{4\pi E(t)h^3}{3a^2k(1-\nu^2)} \tag{10}$$

Furthermore, we assume that during the time interval from impact to fracture the viscoelastic behavior of the plate material may be adequately described by the modified Maxwell element model such as

$$E(t)=E_f+(E_0-E_f)e^{-t/t_R} \tag{11}$$

where E_0 is the instaneous modulus, E_f is the equilibrium modulus and t_R is a principal relaxation time.

Then

$$\bar{G}(s)=\frac{4\pi h^3\bar{E}(s)}{3a^2k(1-\nu^2)}$$

To simplify results even further, we suppose that $q(t)$ is negligible compared to $P(t)$ and hence ignore it in evaluating equations (5), (6), and (7). Thus

$$P(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{mv_0B^2(\alpha/t_R+s)e^{st}ds}{s^3+1/t_Rs^2+B^2s+B^2/t_R} \tag{12}$$

where $B^2=\frac{4\pi h^3E_0}{3a^2k(1-\nu^2)m}$, $\alpha=E_0/E_f$

Note that denominator in (12) is a cubic polynomial of s. Thus we can write

$$P(t)=mv_0B^2(a_1e^{s_1t}+a_2e^{s_2t}+a_3e^{s_3t}) \tag{13}$$

where

$$a_i=\lim_{s\to s_i}\frac{\alpha/t_R+s}{\left[\frac{(s-s_1)(s-s_2)(s-s_3)}{(s-s_i)}\right]}$$

and each s_i, $i=1,2,3$ is a root of the cubic polynomial of denominator in (12). Finally the fracture initiation energy becomes

$$U_c=\frac{1}{2}mv_0^2\left[1-\left\{1+B^2\left[\frac{a_1}{s_1}(1-e^{s_1t_c})+\frac{a_2}{s_2}(1-e^{s_2t_c})+\frac{a_3}{s_3}(1-e^{s_3t_c})\right]\right\}^2\right] \tag{14}$$

where t_c is the smallest positive solution of $P(t_c)=P_c$.

It might prove useful to have an estimate for P_c in terms of a critical uniform biaxial stress level at the surface of the plate opposite the impact point.

For this purpose, we observe that the maximum tensile stress opposite a concentrated load P at the center of an elastic circular is given by

$$\sigma_{max}=\frac{\gamma P}{h^2} \tag{15}$$

where $\gamma=(1+\nu)(.485ln\frac{a}{h}+.52)+\gamma'$ end $\gamma'=0$ for a clamped boundary while $\gamma'=.48$ for a simply supported boundary. Again we suppose that between impact and fracture the quasi-static equilibrium stresses under the load for the viscoelastic plate are reasonably well approximated by those associated with the elastic solution so that our crack initiation criterion can be written:

$$P_c=\frac{h^2S_c}{\gamma} \tag{16}$$

where S_c is a critical stress level.

Of particular interest is the dependence of the fracture initiation energy U_c on the impact velocity v_0 the mass of impacting dart m, and the critical stress S_c. For definiteness we adopt as typical data for a TPP Dymetron disc sample the following values:

$E_0=2.9\times10^5$psi (1.999Gpa)

$E_f=1.16\times10^5$psi (0.799Gpa)

$\nu=.3$

$t_R=.029$sec

$a=1.5$in (0.0381m)

$h=.125$in (0.00318m)

In addition, as standard conditions we take the weight of the dart as $W=4.8$ lb (4.448N) (the weight of the dart-piston assembly in the Dymetron), the impact velocity $v_0=15$ ft/sec (4.572m/s) and the critical stress $S_c=20,000$psi (0.138Gpa). We have also assumed simply supported conditions at the disc boundary. This corresponds to a fracture initiation energy of $U_c=1.227$ ft-lb(1.664N-m) and a fracture initiation time of $t_c=1.436$ms.

In Fig. 2 we have plotted the effect of varying impact velocity on fracture initiation energy values for three values of the dart mass around the standard conditions. The fracture initiation energy appears to be relatively insensitive to the kinetic energy delivered which indicates that both instrumented falling weight and Dymetron tests should correlate reasonably well with conventional falling weight results.

In Fig. 3 we show the effect of variations in critical stress which account for the rather large scatter frequently seen in (unnotched) impact strength data, as the critical stress would be quite sensitive to surface conditions. The corresponding fracture initiation time can be seen in Fig. 4.

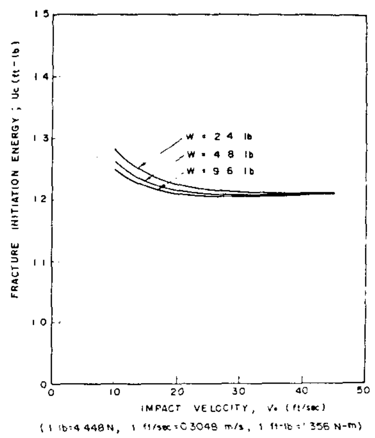

Fig. 2 Fracture initiation energy vs. impact velocity

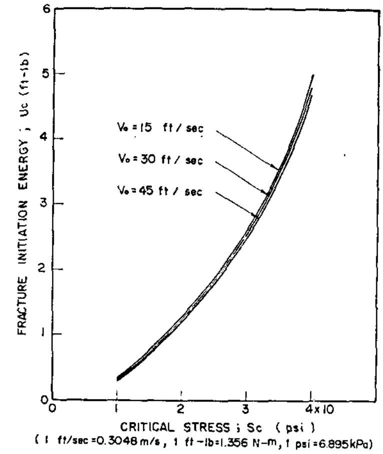

Fig. 3 Fracture initiation energy vs. Critical stress

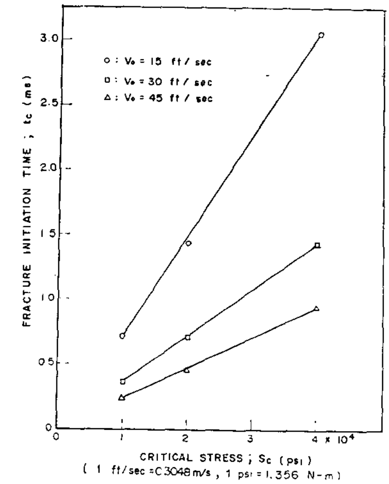

Fig. 4 Fracture initiation time vs. Critical stress

4. Discussion of the Analysis

Several assumptions and simplifications were introduced in the preceding sections which have implications regarding the validity and range of applicability of the results. The basic assumptions underlying the entire analysis is that a quasi-static linear viscoelastic model is appropriate. This would certainly not be the case if stress waves due to the impact had insufficient time to attenuate at fracture initiation.

In a viscoelastic material plane compression stress waves will propagate with a velocity as shown in Christensen[4]

$$C_p=\sqrt{\frac{E_0(1-\nu_0)}{(1+\nu_0)(1-2\nu_0)\rho}} \tag{17}$$

where E_0, ν_0 are the inital values of the uniaxial relaxation modulus and Poisson's ratio and ρ is the mass density of the material. For illustrative purposes we take $E_0=2.9\times10^5$psi (1.999Gpa), $\nu_0=.3$, $\rho=1.728$ slug/ft^3(890.6kg/m^3) (which might correspond to a toughened polypropylene at a temperature of-20F) and find $C_p=5703$ft/sec(1738.3m/s) compared to an impact velocity of about 12ft/sec(3.658m/s) in a typical falling weight test. For a plate thickness of .125 in(0.00318m). it would take the compression wave only .00187 ms to tansverse the thickness of the plate whereas fracture initiation times are typically on the order of 1 to 5 ms.

To estimate the effect of the boundary conditions a conservative measure of the stress wave speed is afforded by the propagation velocity of shear waves;

$$C_s=\sqrt{\frac{\mu_0}{\rho}} \tag{18}$$

where $\mu_0=E_0/2(1+\nu_0)$ is the initial value of the shear relaxation modulus.

For a 3 in (0.0762m). diameter disc of the same material as above the time needed for a shear wave to reflect from the boundary and return to the center of the plate is .082 ms which is still comfortably shorter than the fracture initiation times quoted above.

At higher impact velocities it is possible that the initial compressive shock wave reflecting off the bottom of the plate as a tension wave of double amplitude could do sufficient damage to reduce considerably the stress level at which fracture is initiated. For a plane shock wave the stress discontinuity associated with a velocity jump of $[\![v]\!]$ is given by

$$[\![\sigma]\!]=-\rho C_p[\![v]\!]$$

If we ignore any attenuation and treat the initial as a plane shock wave with a velocity jump v_0, the maximum tensile stress on reflection from the face is

$$\sigma=2\rho C_p v_0$$

which ranges from 1642 psi (11.32 Mpa) to 4106 psi (28.31 Mpa) for impact velocities of 12 ft/sec (3.658m/s) to 30 ft/sec (9.144 m/s) in the example material. A typical value of the tensile yield stress is around 4000 psi (27.58 Mpa)

The use of equation (1) to describe the load-deflection behavior of the plate is based on the assumption that the material is viscoelastic since the load is uniquely determined by the history of the deflection of the plate (again, ignoring inertia effects). Then, under very general conditions the form of equation (1) is a good approximation for "small" motions as shown in Truesdell[5]. Furthermore, if the material behavior is linear with constant Poisson's ratio and the deformations small, then within the bounds of elementary quasi-static plate theory the normal stress distribution in linear through the plate thickness and the plate modulus is related to the uniaxial tension modulus through

$$D(t)=\frac{E(t)h^3}{12(1-\nu^2)}$$

Next we take up the fracture initiation criterion equation (2) which rests on the following ideas. The basic assumption of linear elastic fracture mechanics is that for a given pattern of loading a crack will propagate when the stress intensity factor at the site reaches a critical value. If flaws at which a crack might initiate are uniformly distributed thorughout the plate, then for a given flaw distribution and loading pattern the stress intensity factor at any site is proportional to the load, and thus a crack will start where the stress is maximum(opposite the load) when the load reaches a critical value. Some modification in this conclusion is indicated if the material is indeed viscoelastic rather than elastic. In particular, we should expect that the onset of crack propagation would depend on the history of the stress intensity factor at the site. The effect of this on the results obtained in the preceding section can be estimated within the frame work of a recent theory of crack initiation and growth in viscoelastic materials due to Shapery[6]. The basic idea is that there is a small zone in the neighborhood of the crack tip (called the failure zone) in which separation has occured but significant forces continue to act between material in adjacent "crack surfaces". The energy required to separate surfaces to the point where these failure zone forces are no longer significant is taken to be material property and hence may be used as a criterion for crack propagation.

5. Fracture Initiation by Shapery's Criterion

By making a series of simplifying assumptions in a rather general but complex analysis, Shapery[6] is able to calculate the time to initiate a fracture in a plane strain situation from the relation

$$\Gamma=\frac{1}{8}K^2(t_i)C^*(t_i) \tag{19}$$

where t_i is the time to initiate crack growth, Γ is the specific energy parameter associated with crack growth in the material, $K(t)$ is the opening mode stress intensity factor at the critical flaw (assumed to be monotone non-decreasing) and $C^*(t)$ is the so called secant compliance of the material defined by

$$C^*(t)=\frac{1}{K^2(t)}\int_0^t C(t-t')\ \frac{d}{dt'}\left[K^2(t')\right]dt' \tag{20}$$

where $C(t)$ is the usual creep compliance. For the proposed modified Maxwell material as in the earlier approximate computations

$$C(t)=\frac{1}{E_f}\left[1+\left(\frac{E_f}{E_0}-1\right)e^{-(E_R/E_0)t/t_R}\right] \tag{21}$$

If we suppose that the stress intensity factor is proportional to the maximum stress and therefore the load $P(t)$ then from equations (13) and (15) we conclude

$$K(t)=K_0(a_1e^{s_1t}+a_2e^{s_2t}+a_3e^{s_3t})$$
$$=K_0F(t) \tag{22}$$

where K_0 is a constant factor. Then we note that the function of time

$$\frac{C^*(t)}{C_0}=\frac{1}{K^2(t)}\int_0^t\frac{C(t-t')}{C_0}\ \frac{d}{dt'}\left[K^2(t')\right]dt'$$
$$=\frac{1}{F^2(t)}\int_0^t\frac{C(t-t')}{C_0}\ \frac{d}{dt'}\left[F^2(t')\right]dt' \tag{23}$$

where

$$\lim_{t\to 0}\ C^*(t)=C_0=\lim_{t\to 0}\ C(t) \tag{24}$$

Substituting equations (22) and (23) into the fracture initiation criteria (19) yields

$$\Gamma=\frac{1}{8}K_0^2F^2(t_i)C^*(t_i)$$

or

$$F(t_i)\sqrt{\frac{C^*(t_i)}{C_0}}=\sqrt{\frac{8\Gamma}{K_0^2C_0}} \tag{25}$$

To relate this criterion to equation (16) we rewrite equation (16) in the form

$$P_c=P(t_c)=mv_0B^2F(t_c)\quad\text{or}$$
$$F(t_c)=\frac{h^2S_c}{mv_0B^2\gamma} \tag{26}$$

We note first that the right hand sides of equations (25) and (26) are constant. In the limiting case when both t_i and t_c approach zero the left hand sides coincide hence the constants are the same and we conclude that t_i and t_c are related to each other through

$$F(t_c)=F(t_i)\sqrt{\frac{C^*(t_i)}{C_0}} \tag{27}$$

Therefore the time to fracture can be expressed as

$$F_c^2=\int_0^{t_i}\frac{C(t-t')}{C_0}\ \frac{d}{dt'}\left[F^2(t')\right]dt' \tag{28}$$

Consequently, substituting (21) into (28), we

Table 1 Comparison of the fracture initiation with Shapery's criterion

S_c(psi)	v_0(ft/s)	t_c(ms)	U_c(ft-lb)	t_i(ms)	U_i(ft-lb)	t_c(ms)	U_c(ft-lb)	t_i(ms)	U_i(ft-lb)
1,000	15	0.7091	0.3055	0.7062	0.3030	0.7056	0.3035	0.7027	0.3010
	30	0.3516	0.3030	0.3508	0.3018	0.3507	0.3020	0.3500	0.3008
	45	0.2338	0.3022	0.2335	0.3014	0.2334	0.3015	0.2331	0.3007
20,000	15	1.4507	1.2434	1.4379	1.2224	1.4356	1.2265	1.4234	1.2062
	30	0.7091	1.2222	0.7062	1.2122	0.7056	1.2141	0.7027	1.2043
	45	0.4700	1.2154	0.4687	1.2089	0.4685	1.2102	0.4672	1.2037
40,000	15	3.1321	5.1837	3.0625	4.9854	3.0529	5.0219	2.9902	4.8422
	30	1.4507	4.9738	1.4379	4.8897	1.4356	4.9059	1.4234	4.8251
	45	0.9581	4.9163	0.9465	4.8621	0.9455	4.8727	0.9403	4.8200

(1psi=6.895kpa, 1ft/s=0.3048m/s, 1ft-lb=1.356N-m)

obtain

$$F_c^2=\frac{1}{\alpha}\int_0^{t_i}\left[1+(\alpha-1)e^{-\alpha(t_i-t')/t_R}\right]\frac{d}{dt'}[F^2(t')]dt' \qquad (29)$$

Now, the problem is posed to find the smallest positive root of this integral equation(29). Even though there is some difficulty for very small α which becomes the case of Maxwell two element model, the analysis is tractable numerically.

Table 1 shows the numerical results including Maxwell two element model for the sample. In the standard case treated in the preceding section we had $t_c=1.436$ ms which corresponds $t_i=1.423$ ms. This in turn decreases the energy to initiate fracture by about 1.7 percent. It can be also observed that the fracture initiation energy for the three element model can be reduced to 1~3 percent than that for Maxwell model.

6. Concluding Remarks

Of particular interest is the dependence of the fracture initiation energy on the impact velocity, the mass of impacting dart and the critical stress. We showed the effect of variations in critical stress which could account for the rather large scatter frequently seen in impact strength data, as the critical stress would be quite sensitive to surface conditions. The fracture initiation energy appeared to be relatively insensitive to the kinetic energy delivered which indicated that both instrumented falling weight and Dymetron tests should correlate reasonably well with conventional falling weight results.

A final consideration concerns the relation between the energy transferred to the sample up to the time fracture is initiated and the additional energy absorbed from the impacting device while cracks propagate through the sample. An analysis of the viscoelastic crack propagation problem under the loading conditions of a "falling dart" test and within the framework of Shapery's theory is all but hopelessly intractable. Nevertheless, we can make some general estimates of the order of magnitude of this additional energy compared to the fracture initiation energy.

In simpler loading situations where the stress intensity factor is high, many investigators including Knauss[7] have calculated and measured crack propagations velocities in viscoelastic sheets. The speeds approach an appreciable fraction of the stress wave propagation velocity. On the basis of the preceding example calculations of fracture initiation time and corresponding wave speeds, it is likely that the time for the cracks to propagate through the sample is negligible compared to the time to initiate fracture, and hence the additional work done by the impacting dart must be insignificant.

More direct evidence of this assertion is reported in Gonzalez and Stowell[2] where the load-time trace for an instrumented falling weight test is reproduced. For samples which cracked in a brittle fashion the rise time of the load is approximately 4 ms which corresponds to the fracture initiation time. The load then dropped to zero sharply (less than .1 ms) indicating completion of the fracture process.

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