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FOURIER SERIES OF A STOCHASTIC PROCESS $X(t, \omega) \in L^2_{s.a.p.}$

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1. Introduction

Throughout this paper, $(\Omega, \mathcal{F}, P)$ is the underlying probability space and, without otherwise mentioned, $X(t, \omega)$, $t \in \mathbf{R}$, is a complex valued stochastic process of the second order, where ω is an element of Ω, that is

$$E|X(t, \omega)|^2 = \|X(t, \cdot)\|^2 < \infty \text{ for every } t.$$

Suppose that $X(t, \omega)$ is measurable on $\mathbf{R} \times \Omega$ and also suppose that

$$\int_a^b \|X(t, \cdot)\|^2 dt < \infty, \text{ for every finite } a < b.$$

In this case, $X(t, \omega)$ is of $L^2(a, b)$ as a function of t almost surely.

DEFINITION 1.1. A stochastic process $X(t, \omega)$ which is continuous in t for almost all ω belongs to $L^2_{u\ a.p.}$ if and only if for the set

$$E\{\varepsilon, X\} \equiv \{\tau : \sup_{t \in R} \|X(t+\tau, \cdot) - X(t, \cdot)\|^2 < \varepsilon\},$$

for every $\varepsilon > 0$, there exists $l = l(\varepsilon)$ such that for every $\alpha \in \mathbf{R}$,

$$(\alpha, \alpha + l) \cap E\{\varepsilon, X\} \neq \phi.$$

DEFINITION 1.2. $X(t, \omega) \in L^2_{s.a.p.}$ if and only if for the set

$$S^2\{\varepsilon, X\} \equiv \left\{\tau : \sup_{u \in R} \int_u^{u+1} \|X(t+\tau, \cdot) - X(t, \cdot)\|^2 dt < \varepsilon\right\},$$

for every $\varepsilon > 0$, there exists $l = l(\varepsilon)$ such that for every $\alpha \in \mathbf{R}$,

$$(\alpha, \alpha + l) \cap S^2\{\varepsilon, X\} \neq \phi.$$

REMARK 1.1. It is known (6) that for a weakly stationary process, $L^2_{u.a.p.}$ and $L^2_{s.a.p.}$ are the same class. For this property it is necessary and sufficient that the covariance function is uniformly almost periodic.

In this paper, we find the Fourier series of $X(t, \omega) \in L^2_{s.a.p.}$ and the Parseval relation of $X(t, \omega) \in L^2_{s.a.p.}$ In section 2, we investigate some basic properties of $X(t, \omega) \in L^2_{s.a.p.}$ In section 3, we show that the mean of $X(t, w) \in L^2_{s.a.p.}$ exists and in section 4, after showing the existence of Fourier exponents and Fourier coefficients of $X(t, \omega) \in L^2_{s.a.p.}$ we give the Parseval relation of $X(t, \omega) \in L^2_{s.a.p.}$

For convenience we will denote $X(t, \omega)$ as $X(t)$ in what follows.

2. Some basic properties of the class $L^2_{s.a.p.}$

We shall give some propositions of $X(t)\in L^2_{s.a.p.}$ whose proofs are omitted.

PROPOSITION 2.1. *Let* $X(t)\in L^2_{s.a.p.}$ *Then*

$$\sup_{u\in R}\int_u^{u+1}\|X(t)\|^2dt<\infty.$$

PROPOSITION 2.2. *Let* $X(t)\in L^2_{s.a.p.}$ *Then*

$$\sup_{u\in R}\int_u^{u+1}\|X(t+h)-X(t)\|^2dt\longrightarrow 0\ \ (as\ h\to 0).$$

Now we will introduce the notion of the normality of $X(t)\in L^2_{s.a.p.}$ which is similar to the normality of $u.a.p.$ functions. This definition enables us to treat $X(t)\in L^2_{s.a.p.}$ as processes possessing this property.

DEFINITION 2.1. $X(t)$ is S^2-normal if and only if given any countable infinite set $\{h_i\}$ of real numbers, for every $\varepsilon>0$, there exists infinite subset $H\subset\{h_i\}$ such that for every pair $h_i, h_j\in H$

$$\sup_{u\in R}\int_u^{u+1}\|X(t+h_i)-X(t+h_j)\|^2dt<\varepsilon$$

PROPOSITION 2.3. *The following statements are equivalent.*

(i) $X(t)$ *is* S^2*-normal.*

(ii) $X(t)\in L^2_{s.a.p.}$

Proof. (ii)⇒(i). Given $\{h_i\}\subset\mathbf{R}$. For each h_i, there exist $\tau_i\in S^2\{\varepsilon/16,\ X\}$ and $0\leq\gamma_i\leq l(=l(\varepsilon/16))$ such that $h_i=\tau_i+\gamma_i$. For each h_i we consider only one representation in this form. Let γ be a limit point of the sequence of all γ_i. Choose $\delta>0$ such that, for $|h|<2\delta$,

$$\sup_{u\in R}\int_u^{u+1}\|X(t+h)-X(t)\|^2dt<\frac{\varepsilon}{4}.$$

Define the set H of all h_i for which $\gamma-\delta<\gamma_i<\gamma+\delta$.
For every pair $h_j,\ h_k\in H$

$$\begin{aligned}
&\sup_{u\in R}\int_u^{u+1}\|X(t+h_j)-X(t+h_k)\|^2dt\\
&=\sup_{u\in R}\int_u^{u+1}\|X(t+\tau_j+\gamma_j)-X(t+\tau_k+\gamma_k)\|^2dt\\
&=\sup_{u\in R}\int_u^{u+1}\|X(t+\tau_j-\tau_k+\gamma_j-\gamma_k)-X(t)\|^2dt\\
&=\sup_{u\in R}\int_u^{u+1}\|X(t+\tau_j-\tau_k)-X(t+\gamma_j-\gamma_k)\|^2dt\\
&\leq 2\sup_{u\in R}\int_u^{u+1}\|X(t+\tau_j-\tau_k)-X(t)\|^2dt\\
&\quad+2\sup_{u\in R}\int_u^{u+1}\|X(t)-X(t+\gamma_j-\gamma_k)\|^2dt\\
&<\varepsilon.
\end{aligned}$$

(i) $\Rightarrow$ (ii). Suppose that $X(t)\notin L^2_{s.a.p.}$

Take now an arbitrary real number h_1. For h_1 there exists $(a_2, b_2)\subset\mathbf{R}$ such that $b_2-a_2>2|h_1|$ and $(a_2, b_2)\cap S^2\{\varepsilon, X\}=\phi$. If we let h_2 be the center of (a_2, b_2) then $h_2-h_1\in(a_2, b_2)$ and therefore $(h_2-h_1)\notin S^2\{\varepsilon, X\}$. There exists $(a_3, b_3)\subset\mathbf{R}$ such that $b_3-a_3>2(|h_1|+|h_2|)$ and $(a_3, b_3)\cap S^2\{\varepsilon, X\}=\phi$. Let h_3 be the center of (a_3, b_3). Then we have h_3-h_1, $h_3-h_2\notin S^2\{\varepsilon, X\}$. For the same reason as before we can take h_4, h_5, ... such that none of the numbers h_i-h_j belongs to $S^2\{\varepsilon, X\}$. Thus for any pair i, j

$$\sup_{u\in R}\int_u^{u+1}\|X(t+h_i)-X(t+h_j)\|^2dt\geq\varepsilon.$$

It is a contradiction to the assumption that $X(t)$ is S^2-normal. Thus we have the conclusion.

PROPOSITION 2.4. (i) *If* $X(t)\in L^2_{s.a.p.}$ *then* $c\cdot X(t)\in L^2_{s.a.p.}$ *for* $c\in C$.
(ii) *If* $X(t)\in L^2_{s.a.p.}$ *then* $\|X(t)\|^2$ *is a real valued s.a.p. function.*

PROPOSITION 2.5. *Let* $X_1(t)$, $X_2(t)\in L^2_{s.a.p.}$ *Then*

$$(X_1+X_2)(t)\overline{\overline{\leftarrow}}X_1(t)+X_2(t)\in L^2_{s.a.p.}$$

PROPOSITION 2.6. *Let* $X_1(t)$, $X_2(t)\in L^2_{s.a.p.}$ *Then*

$$\langle X_1,\ X_2\rangle(t)\overline{\overline{\leftarrow}}\langle X_1(t),\ X_2(t)\rangle$$

is a real valued s.a.p. function.

3. Fourier series

To construct Fourier series of $X(t)\in L^2_{s.a.p.}$, first we show that the mean of $X(t)\in L^2_{s.a.p.}$ exists, next we give some propositions.

THEOREM 3.1. *Let* $X(t)\in L^2_{s.a.p.}$ *Then there exists*

$$\underset{T\to\infty}{\text{l.i.m.}}\frac{1}{T}\int_\alpha^{\alpha+T}X(t)dt,\ \textit{independent for }\alpha.$$

(*For the integral,* see I.I. Gikhman, and A.V. skorokhod, (3) page 185).

Proof. Since $X(t)\in L^2_{s.a.p.}$, for every $\varepsilon>0$, there exists $L=l(\varepsilon)>0$ such that $(\alpha, \alpha+L)\cap S^2\{\varepsilon, X\}\neq\phi$ for every $\alpha\in\mathbf{R}$.

Let
$$I=\|\frac{1}{T}\int_0^T X(t)dt-\frac{1}{T}\int_\alpha^{\alpha+T}X(t)dt\|.$$

Then for $\tau\in S^2\{\varepsilon,\ X\}$,

$$\begin{aligned}I&\leqq\left\|\frac{1}{T}\int_0^T X(t)dt-\frac{1}{T}\int_\tau^{\tau+T}X(t)dt\right\|+\left\|\frac{1}{T}\int_\alpha^\tau X(t)dt\right\|+\left\|\frac{1}{T}\int_{\alpha+T}^{\tau+T}X(t)dt\right\|\\&\leqq\left[\int_0^T\|X(t)-X(t+\tau)\|^2dt\right]^{\frac{1}{2}}+\frac{L}{T}\left[\int_\alpha^\tau\|X(t)\|^2dt\right]^{\frac{1}{2}}+\frac{L}{T}\left[\int_{\alpha+T}^{\tau+T}\|X(t)\|^2dt\right]^{\frac{1}{2}}\end{aligned}$$

$$<\left[([T]+1)\varepsilon\right]^{\frac{1}{2}}+\frac{2L}{T}\left[\sup_{\alpha\in R}\int_{\alpha}^{\alpha+L}\|X(t)\|^2dt\right]^{\frac{1}{2}}$$

$$<c_1\sqrt{\varepsilon}+c_2\frac{1}{T} \text{ for some constants } c_1, c_2.$$

Therefore, for every $\varepsilon>0$, there exists $T_0(\varepsilon)$ such that for any $T\geq T_0(\varepsilon)$

$$\left\|\frac{1}{T}\int_0^T X(t)\,dt-\frac{1}{T}\int_{\alpha}^{\alpha+T}X(t)\,dt\right\|<\varepsilon, \text{ independent for } \alpha.$$

Let $\alpha=(v-1)T$, $v=1,2,\ldots,n$, then

$$\left\|\frac{1}{T}\int_0^T X(t)\,dt-\frac{1}{T}\int_{(v-1)T}^{vT}X(t)\,dt\right\|<\varepsilon.$$

Therefore, for any $T>T_0(\varepsilon)$, we have

$$\left\|\frac{1}{T}\int_0^T X(t)\,dt-\frac{1}{nT}\int_0^{nT}X(t)\,dt\right\|<\varepsilon.$$

Hence for T_1, $T_2>T_0(\varepsilon)$

$$\begin{aligned}&\left\|\frac{1}{T_1}\int_0^{T_1}X(t)\,dt-\frac{1}{T_2}\int_0^{T_2}X(t)\,dt\right\|\\ &\leqq\left\|\frac{1}{T_1}\int_0^{T_1}X(t)\,dt-\frac{1}{n_1T_1}\int_0^{n_1T_1}X(t)\,dt\right\|\\ &+\left\|\frac{1}{T_2}\int_0^{T_2}X(t)\,dt-\frac{1}{n_2T_2}\int_0^{n_2T_2}X(t)\,dt\right\|\\ &+\left\|\frac{1}{n_1T_1}\int_0^{n_1T_1}X(t)\,dt-\frac{1}{n_2T_2}\int_0^{n_2T_2}X(t)\,dt\right\|\\ &\leq 3\varepsilon.\end{aligned}$$

PROPOSITION 3.1. *Let $X(t)\in L^2_{s.a.p.}$ Then there exists*

$$\varphi(u)=\lim_{T\to\infty}\frac{1}{T}\int_{\gamma}^{\gamma+T}\langle X(t+u),\ X(t)\rangle\,dt.$$

(convergence of the above limit is uniform for u and independent for γ).

Proof. From Proposition 2.6. $\langle X(t+u), X(t)\rangle$ is a real valued s.a.p. function with respect to t. Let $\varphi_u(t)\equiv\langle X(t+u),\ X(t)\rangle$.
Similarly in Theorem 3.1., we have

$$\begin{aligned}&\left|\frac{1}{T}\int_0^T\varphi_u(t)\,dt-\frac{1}{T}\int_{\gamma}^{\gamma+T}\varphi_u(t)\,dt\right|\\ &\leq\frac{1}{T}\int_0^T|\varphi_u(t)-\varphi_u(t+\tau)|\,dt+\frac{1}{T}\int_{\gamma}^{\tau}|\varphi_u(t)|\,dt+\frac{1}{T}\int_{\gamma+T}^{\tau+T}|\varphi_u(t)|\,dt\\ &=I_1+I_2+I_3 \text{ (say)},\end{aligned}$$

where $\tau\in S^2\{\varepsilon, X\}$. τ does not depend on u.
By the following relation

$$\begin{aligned}&|\varphi_u(t)-\varphi_u(t+\tau)|\\ &=|\langle X(t+u),\ X(t)\rangle-\langle X(t+\tau+u),\ X(t+\tau)\rangle|\\ &\leqq\|X(t+u)-X(t+\tau+u)\|\ \|X(t)\|+\|X(t+\tau+u)\|\ \|X(t)-X(t+\tau)\|\end{aligned}$$

we have

$$I_1=\frac{1}{T}\int_0^T|\varphi_u(t)-\varphi_u(t+\tau)|dt$$
$$\leq\left\{\frac{1}{T}\int_0^T\|X(t+u)-X(t+\tau+u)\|^2dt\right\}^{\frac{1}{2}}\cdot\left\{\frac{1}{T}\int_0^T\|X(t)\|^2dt\right\}^{\frac{1}{2}}$$
$$+\left\{\frac{1}{T}\int_0^T\|X(t+\tau+u)\|^2dt\right\}^{\frac{1}{2}}\cdot\left\{\frac{1}{T}\int_0^T\|X(t)-X(t+\tau)\|^2dt\right\}^{\frac{1}{2}}$$
$$<\left\{\frac{[T]+1}{T}\sup_{l\in R}\int_l^{l+1}\|X(t+u)-X(t+\tau+u)\|^2dt\right\}^{\frac{1}{2}}\cdot\left\{\frac{[T]+1}{T}\sup_{l\in R}\int_l^{l+1}\|X(t)\|^2dt\right\}^{\frac{1}{2}}$$
$$+\left\{\frac{[T]+1}{T}\sup_{l\in R}\int_l^{l+1}\|X(t+\tau+u)\|^2dt\right\}^{\frac{1}{2}}\cdot\left\{\frac{[T]+1}{T}\sup_{l\in R}\int_l^{l+1}\|X(t)-X(t+\tau)\|^2dt\right\}^{\frac{1}{2}}$$
$$\leq c_2\sqrt{\varepsilon},\ \text{for some constant } c_2.$$

And

$$I_3=\frac{1}{T}\int_{\gamma+T}^{\tau+T}|\varphi_u(t)|dt$$
$$\leq\frac{1}{T}\int_{\gamma+T}^{\tau+T}\|X(t+u)\|\ \|X(t)\|dt$$
$$<\frac{1}{T}\sup_{\gamma\in R}\int_\gamma^{\gamma+L}\|X(t)\|^2dt$$
$$\leq\frac{c_3}{T},\ \text{for some constant } c_3.$$

Similarly we have

$$I_2\leq\frac{c_3}{T}.$$

Hence

$$\left|\frac{1}{T}\int_0^T\varphi_u(t)dt-\frac{1}{T}\int_\gamma^{\gamma+T}\varphi_u(t)dt\right|$$
$$<c_2\sqrt{\varepsilon}+2c_3\frac{1}{T}.$$

Therefore for every $\varepsilon>0$, there exists $T_0=T_0(\varepsilon)$ such that for every $T\geq T_0(\varepsilon)$,

$$\left|\frac{1}{T}\int_0^T\varphi_u(t)dt-\frac{1}{T}\int_\gamma^{\gamma+T}\varphi_u(t)dt\right|<\varepsilon.$$

T_0 does not depend on γ and u. And using similar argument of Theorem 3.1., we get

$$\lim_{T\to\infty}\frac{1}{T}\int_\gamma^{\gamma+T}\langle X(t+u),\ X(t)\rangle dt=\varphi(u).$$

Convergence of left hand side is uniform for u and independent for γ.

PROPOSITION 3.2. *Let $X(t)\in L^2_{s.a.p.}$ Then $\varphi(u)$ is a real valued $s^2.a.p.$ function.*

Proof. For every $l\in R$ and for $\tau\in S^2\{\varepsilon, X\}$,

$$\int_l^{l+1}|\varphi(u)-\varphi(u+\tau)|^2du$$

$$=\int_l^{l+1}\left|\lim_{T\to\infty}\frac{1}{T}\int_0^T\langle X(u+t)-X(u+\tau+t),\quad X(t)\rangle dt\right|^2 du$$
$$\leq\varlimsup_{T\to\infty}\int_l^{l+1}\left\{\frac{1}{T}\int_0^T\|X(u+t)-X(u+\tau+t)\|^2dt\right\}\left\{\frac{1}{T}\int_0^T\|X(t')\|^2dt'\right\}du$$

By propoistion 2.1., this is not greater than

$$c_4\varlimsup_{T\to\infty}\frac{1}{T}\int_0^T\int_l^{l+1}\|X(u+t)-X(u+\tau+t)\|^2du\ dt$$
$$<c_4\,\varepsilon,\ \text{for some constant } c_4.$$

Therefore for $\tau\in S^2\{\varepsilon,\ X\}$, τ belongs to

$$S'\{c_4\,\varepsilon,\ \varphi\}\equiv\left\{\tau:\sup_{l\in R}\int_l^{l+1}|\varphi(u)-\varphi(u+\tau)|^2du<\varepsilon\right\}.$$

PROPOSITION 3.3. *Let $X(t)\in L^2_{s.a.p.}$ Then $\varphi(u)$ is non negative definite function.*

i.e. (i) *$\varphi(u)$ is continuous at $u=0$.*

(ii) $\sum_{i=1}^n\sum_{j=1}^n\xi_i\bar{\xi}_j\varphi(u_i-u_j)\geq 0$

for all choices of finite subsets $(u_1,\ \dots,\ u_n)$ of u and n-tuples $(\xi_1,\ \dots,\ \xi_n)$ of complex numbers.

Proof. (i) $|\varphi(u)-\varphi(0)|$

$$=\left|\lim_{T\to\infty}\frac{1}{T}\int_0^T\langle X(u+t)-X(t),\ X(t)\rangle dt\right|$$
$$\leq c_5\left\{\sup_{l\in R}\int_l^{l+1}\|X(u+t)-X(t)\|^2dt\right\}^{\frac{1}{2}}\longrightarrow 0\ \ (\text{as } u\to 0),\ \text{for some constant}$$

c_5.

(ii) $\sum_{i=1}^n\sum_{j=1}^n\xi_i\bar{\xi}_j\varphi\ (u_i-u_j)$

$$=\sum_{i=1}^n\sum_{j=1}^n\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^T\langle\xi_iX(u_i-u_j+t),\ \xi_jX(t)\rangle dt$$
$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^T\left\|\sum_{i=1}^n\xi_iX(u_i+t)\right\|^2dt$$
$$\geq 0.$$

REMARK 3.1. Form Proposition 3.3. (i), we have $\varphi(u)$ is uniformly continuous and therefore $\varphi(u)$ is a real valued u.a.p. function (A.S. Besicovitch, (I)).

4. Parseval relation

After showing the existence of Fourier exponents and Fourier coefficients of $X(t)\in L^2_{s.a.p.}$, we shall give Parseval relation in the case of $X(t)\in L^2_{s.a.p.}$

PROPOSITION 4.1. *Let $X(t)\in L^2_{s.a.p.}$ Then there exists $\{\lambda_n\}\subset\mathbf{R}$ and $\{\gamma_n\}\subset\mathbf{R}^+$ such that*

$$\sum_{n=1}^\infty\gamma_n<\infty,\quad\varphi(u)=\sum_{n=1}^\infty\gamma_ne^{i\lambda_nu}.$$

Proof. Since $\varphi(u)$ is non negative definite, by using Bochner's theorem (*H.* Cramér, and H.R. Leadbetter, (2), page 126) there exists $F(\lambda)$ which is real, nondecreasing, and bounded such that

$$\varphi(u)=\int_{-\infty}^{\infty} e^{i\lambda u}dF(\lambda).$$

Let $\gamma(\lambda)=F(\lambda+0)-F(\lambda-0)$ for every $\lambda\in\{\lambda_1, \lambda_2, \ldots\}=\Lambda$ (it may be ϕ) and for $\lambda\notin\Lambda$, $\gamma(\lambda)=0$. And

$$\gamma(\lambda_j)\equiv\gamma_j\geq 0,\quad \sum_{j=1}^{\infty}\gamma_j\leq F(\infty)-F(-\infty)<\infty,$$
$$F_1(\lambda)\equiv\sum_{\lambda j\leq\lambda}\gamma_j,\quad F_2(\lambda)\equiv F(\lambda)-F_1(\lambda).$$

Then we have

$$\varphi(u)=\sum_{n=1}^{\infty}\gamma_n e^{i\lambda_n u}+\int_{-\infty}^{\infty} e^{i\lambda u}dF_2(\lambda).$$

From the facts that $\varphi(u)$ is u. a. p. and

$$\sum_{n=1}^{\infty}\gamma_n e^{i\lambda_n u} \text{ is u. a. p.,}$$

we have $\varphi_2(u)=\int_{-\infty}^{\infty} e^{i\lambda u}\, dF_2(\lambda)$ is u. a. p. and therefore $|\varphi_2(u)|^2$ is also u. a. p. And then

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_0^T|\varphi_2(u)|^2du\\
&=\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\lambda-\lambda')u}dF_2(\lambda)dF_2(\lambda')\Big)du\\
&=\int_{-\infty}^{\infty}(F_2(\lambda^+)-F_2(\lambda^-))dF_2(\lambda)\\
&=0.
\end{aligned}$$

From the above relation we get $\varphi_2(u)=0$. (A.S. Besicovitch, (I))

Therefore $\varphi(u)=\sum_{n=1}^{\infty}\gamma_n e^{i\lambda_n u}$.

THEOREM 4.1. *Let* $X(t)\in L^2_{s.a.p.}$
For $\alpha(\lambda)=\underset{T\to\infty}{\text{l.i.m.}}\ \frac{1}{T}\int_0^T X(t)e^{-i\lambda t}dt$,
there exists $\Lambda=\{\lambda_n\}\subset\mathbf{R}$ *such that* $\alpha(\lambda)\neq 0$ *for* $\lambda\in\Lambda$ *and* $\alpha(\lambda)=0$ *for* $\lambda\notin\Lambda$.
Let $\alpha(\lambda_n)\equiv\alpha_n$, $n=1, 2, \ldots$. *Then Parseval relation*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\|X(t)\|^2dt=\sum_{n=1}^{\infty}\|\alpha_n\|^2 \textit{ holds.}$$

(*We call the numbers* $\lambda_1, \lambda_2, \ldots,$ *Fourier exponents and the numbers* $\alpha_1, \alpha_2, \ldots,$ *Fourier coefficients of* $X(t)\in L^2_{s.a.p.}$).

Proof. From Proposition 4.1., we know there exist $\Lambda=\{\lambda_n\}$ and $\{\gamma_n\}\subset\mathbf{R}^+$ such that $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\varphi(u)=\sum_{n=1}^{\infty}\gamma_n e^{i\lambda_n u}$. Therefore we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi(u)e^{-i\lambda u}du$$
$$=\lim_{T\to\infty}\frac{1}{T}\int_0^T\sum_{n=1}^{\infty}\gamma_n e^{i(\lambda_n-\lambda)u}du$$
$$=\sum_{n=1}^{\infty}\gamma_n\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i(\lambda_n-\lambda)u}du\right)$$
$$=\begin{cases}\gamma_n & \text{if } \lambda=\lambda_n\in\Lambda, \quad n=1,2,\ldots.\\ 0 & \text{if } \lambda\in\Lambda^c.\end{cases}$$

Otherwise

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi(u)e^{-i\lambda u}du$$
$$=\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-i\lambda u}\left\{\lim_{S\to\infty}\frac{1}{S}\int_0^S\langle X(u+t),\ X(t)\rangle dt\right\}du$$
$$=\lim_{S\to\infty}\langle \underset{T\to\infty}{\text{l.i.m.}}\frac{1}{T}\int_{-t}^{T-t}e^{-i\lambda u}X(u)du,\ \frac{1}{S}\int_0^S e^{-i\lambda t}X(t)dt\rangle$$
$$=\|\alpha(\lambda)\|^2.$$

Hence $\qquad \|\alpha(\lambda_n)\|^2=\|\alpha_n\|^2=\gamma_n,\ \ n=1,\ 2,\ \ldots.$

$\|\alpha(\lambda)\|^2=0,\ \text{ for } \lambda\in\Lambda^c,$

Therefore we have

$$\varphi(0)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\|X(t)\|^2dt.$$
$$=\sum_{n=1}^{\infty}\gamma_n=\sum_{n=1}^{\infty}\|\alpha(\lambda_n)\|^2=\sum_{n=1}^{\infty}\|\alpha_n\|^2.$$

PROPOSITION 4.2. *Let* $X(t)\in L^2_{s.a.p.}$ *Let* $\Lambda=\{\lambda_n\}$, α_n, $n=1,2,\ldots$, *be Fourier exponents and corresponding Fourier coefficients in* Proposition 4.2. *Then*

$$\lim_{N\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T\|X(t)-\sum_{n=1}^{N}\alpha_n e^{i\lambda_n t}\|^2dt=0.$$

Proof. $\displaystyle\lim_{N\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T\|X(t)-\sum_{n=1}^{N}\alpha_n e^{i\lambda_n t}\|^2dt$

$$=\lim_{N\to\infty}\left[\lim_{T\to\infty}\frac{1}{T}\int_0^T\|X(t)\|^2dt-\sum_{n=1}^{N}\|\alpha_n\|^2\right]$$
$$=0$$

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