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ON THE NUMERICAL RANGE FOR NONLINEAR OPERATORS

KYONG SOO KIM AND YOUNGOH YANG

1. Introduction

Let B be a unital C^*-algebra, $B^\#$ the dual space of B, and X the Hilbert B-module with a B-valued inner product $\langle , \rangle$ [6]. We define the norm $\|\cdot\|_X$ on X by $\|x\|_X = \|\langle x, x\rangle\|^{\frac{1}{2}}$. Then for any bounded linear operator T on X, the B-spatial numerical range $W_B(T)$ is defined by $W_B(T) = \{f(\langle Tx, x\rangle) : x \in X, f \in B^\#, \|x\|_X = \|f\| = f(\langle x, x\rangle) = 1\}$ [7]. In [2], Canavati defined a numerical range for the class of all numerically bounded (nonlinear) maps on a Banach space and gave some of the basic properties of such numerical range. In this paper, we shall define a numerical range for a new class of all B^*-numerically bounded maps on a Hilbert B-module, and study analogous results of it in a systematic way. Among other properties, our numerical range will be compact and connected, and will coincide with $\overline{W_B(T)}$, in the particular case when T is a bounded linear operator on a Hilbert B-module X.

Throughout this paper, a Hilbert B-module X is assumed to have a vector space structure over the complex numbers $\mathbf{C}$ compatible with that of B in the sense that $\lambda(xb) = (\lambda x)b = x(\lambda b)$ $(x \in X, b \in B, \lambda \in \mathbf{C})$. We will use the following notations. $L(X)$ is the Banach space of all bounded linear operators on X. $B(X)$ is the vector space of all continuous maps $P : X \longrightarrow X$ such that $\|P(x)\|_X \leqq M\|x\|_X$ for some $M \geqq 0$ and all x in X. An element of $B(X)$ is called a bounded map on X. $Q(X)$ is the vector space of all quasibounded maps on X [2], [3]. We also denote the operator norm on $L(X)$ by $\|\cdot\|$.

2. Some Banach spaces of nonlinear maps.

In [7], the norm$\times$weak* topology in $X \times B^\#$ is defined as the product topology in $X \times B^\#$ given by the norm topology on X and the weak* topology on $B^\#$. We consider the following subsets of $X \times B^\#$. $\Pi_r = \{(x, f) \in X \times B^\# : \|x\|_X = \|f\| \geqq r, f(\langle x, x\rangle) = \|x\|_X{}^3\}$ for $r > 0$, and $\Pi_0 = \bigcup_{r>0} \Pi_r$.

The following two results are essenstially due to Bonsall, Cain and Schneider [1]. We omit the proofs of them, because they would be similar as in [2].

LEMMA. *Let π denote the natural projection of $X \times B^\#$ onto X, and let A be a subset of Π_r that is relatively closed in Π_r with respect to the norm$\times$weak** *topology. Then $\pi(A)$ is a norm closed subset of X.*

PROPOSITION 2.1. *Each $\Pi_r(r>0)$ and Π_0 are connected subsets of $X\times B^{\#}$ with the norm$\times$weak* topology, unless X has dimension one over R.*

From now on we shall assume that Π_0 has the norm$\times$weak* topology induced as a subset of $X\times B^{\#}$. Also we shall assume that X does not have dimension one over R.

DEFINITION 2.1. *A continuous map $F : \Pi_0 \longrightarrow X$ is called B^*-bounded* if $\|F\|^* =\sup_{\Pi_0}\frac{\|F(x,f)\|_X}{\|x\|_X}<\infty$. We denote by $B^*((X)$, the vector space of all B^*-bounded maps. Notice that $\|\cdot\|^*$ is a norm on $B^*(X)$ and $B^*(X)$ is a Banach space. We can consider the vector space $B(X)$ as a vector subspace of $B^*(X)$ in a natural way, namely; if $P\in B(X)$, then the mapping $F(x,f)=P(x)$ belongs to $B^*(X)$ and $\|P\|=\|F\|^*$ [2].

DEFINITION 2.2. *A continuous map $F : \Pi_0 \longrightarrow X$ is called B^*-quasibounded if* $|F|^*=\lim_{r\to\infty}\sup_{\Pi_r}\frac{\|F(x,f)\|_X}{\|x\|_X}<\infty$.
We denote by $Q^*(X)$, the vector space of all B^*-quasibounded maps. Notice that $|\cdot|^*$ is a seminorm on $Q^*(X)$. Obviously one has $B^*(X)\subset Q^*(X)$ and $|F|^*\leqq\|F\|^*$. We can consider the vector space $Q(X)$ as a vector subspace of $Q^*(X)$ in a natural way, namely; if $P\in Q(X)$, then the mapping $F(x,f)=P(x)$ belongs to $Q^*(X)$ and $|P|=|F|^*$. Here $|\cdot|$ is a seminorm on $Q(X)$ [2], [3].

PROPOSITION 2.2. *For any $F\in Q^*(X)$, there exists a sequence $\{F_n\}$ in $B^*(X)$ such that $|F_n-F|^*=0$ $(n=1,2,3,\ldots)$ and $\|F_n\|^*\longrightarrow|F|^*$ as $n\longrightarrow\infty$.*

Proof. Let $\rho^2=\|x\|_X^2+\|f\|^2$, and define $F_n(x,f)=F(x,f)$ if $\rho\geqq n$, $F_n(x,f)=\frac{\rho}{n}F\left(\frac{n}{\rho}x,\frac{n}{\rho}f\right)$ if $0<\rho<n$. We have
$\|F_n\|^*=\sup_{\Pi_0}\frac{\|F_n(x,f)\|}{\|x\|_X}X=\sup_{\Pi_{\frac{n}{\sqrt{2}}}}\frac{\|F(x,f)\|_X}{|x\|_X}$. Therefore $F_n\in B^*(X)$ for all n large enough and $\|F_n\|^*\longrightarrow|F|^*$ as $n\longrightarrow\infty$.

DEFINITION 2.3. *Let F, $G\in Q^*(X)$. The mapping F is said to be B^*-asymptotically equivalent to G if $|F-G|^*=0$.* It is easy to see that this is an equivalence relation.
$\tilde{Q}^*(X)$ is the normed space of all equivalence classes of B^*-quasibounded maps, i.e., $\tilde{Q}^*(X)=Q^*(X)/N(|\cdot|^*)$, where $N(|\cdot|^*)=\{F\in Q^*(X) : |F|^*=0\}$. The norm on $\tilde{Q}^*(X)$ is the one induced by $|\cdot|^*$ and will be denoted in the same way.
From Proposition 2.2, we see that the mapping $B^*(X)\longrightarrow\tilde{Q}^*(X)$, $F\longrightarrow\tilde{F}$ is onto.

PROPOSITION 2.3. *$\tilde{Q}^*(X)$ is a Banach space.*

Proof. Let $\{\tilde{F}_n\}$ be a sequence in $\tilde{Q}^*(X)$ such that $\sum |\tilde{F}_n|^* < \infty$. We have to show that $\sum \tilde{F}_n$ converges. By Proposition 2. 2, for any positive integer n we can choose $G_n \in B^*(X)$ such that $\tilde{G}_n = \tilde{F}_n$ and $\|G_n\|^* \leqq |F_n|^* + 2^{-n}$. Since $B^*(X)$ is a Banach space, $\sum G_n$ converges to an element $G \in B^*(X)$. From the continuity of the linear projection $B^*(X) \longrightarrow \tilde{Q}^*(X)$ we obtain $\sum \tilde{G}_n = \sum \tilde{F}_n = \tilde{G}$.

DEFINITION 2.4. *A continuous map $F: \Pi_0 \longrightarrow X$ is called B^*-numerically bounded* if

$$\omega^*(F) = \lim_{r\to\infty} \sup_{\Pi r} \frac{|f(\langle F(x, f), x\rangle)|}{\|x\|_X{}^2 \ \|f\|} < \infty.$$

We denote by $W^*(X)$, the vector space of all B^*-numerically bounded maps. Notice that ω^* is a seminorm on $W^*(X)$. If $F \in W^*(X)$, then we let

$$\alpha^*(F) = \lim_{r\to\infty} \inf_{\Pi r} \frac{|f(\langle F(x,f), x\rangle)|}{\|x\|_X{}^2 \|f\|}.$$

Obviously one has $Q^*(X) \subset W^*(X)$ and $\omega^*(F) \leqq |F|^*$.

DEFINITION 2.5. *Let $F \in W^*(X)$ and consider the maps $F_\nu : \Pi_0 \longrightarrow X$ and $F_\tau : \Pi_0 \longrightarrow X$ given by $F_\nu(x, f) = \dfrac{f(\langle F(x, f), x\rangle)}{\|x\|_X{}^2 \|f\|} x$ and $F_\tau(x, f) = F(x, f) - F_\nu(x, f)$. Then $F = F_\nu + F_\tau$. The maps F_ν and F_τ are called the normal and tangent components of F respectively.* It is easy to show that if $F \in W^*(X)$, then

(a) $f(\langle F_\nu(x, f), x\rangle) = f(\langle F(x, f), x\rangle)$, $(x, f) \in \Pi_0$.
(b) $f(\langle F_\tau(x, f), x\rangle) = 0$, $(x, f) \in \Pi_0$.
(c) $F_\nu \in Q^*(X)$ and $|F_\nu|^* = \omega^*(F)$. Hence we obtain the following result.

PROPOSITION 2.4. *$F \in W^*(X)$ if and only if there exists continuous mappings $G, H : \Pi_0 \longrightarrow X$ with $G \in Q^*(X)$ and H satisfying $f(\langle H(x,f), x\rangle) = 0$ ($(x,f) \in \Pi_0$) such that $F = G + H$. Such a map H is called a B^*-orthogonal map.*

DEFINITION 2.6. *Let $F, G \in W^*(X)$. The mapping F is said to be B^*-asymptotically numerically equivalent to G* if $\omega^*(F-G) = 0$. It is easy to see that this is an equivalence relation. $\hat{W}^*(X)$ is the normed space of all equivalence classes of B^*-numerically bounded maps, i.e., $\hat{W}^*(X) = W^*(X)/N(\omega^*)$, where $F \in N(\omega^*)$ iff $\omega^*(F) = 0$. The norm on $\hat{W}^*(X)$ is the one induced by ω^*, and it will be denoted in the same way.

PROPOSITION 2.5. *$\hat{W}^*(X)$ is a Banach space.*

Proof. Let $\{\hat{F}_n\}$ be a sequence in $\hat{W}^*(X)$ such that $\sum \omega^*(\hat{F}_n) < \infty$. We have to show that $\sum \hat{F}_n$ converges. Since $\omega^*(\hat{F}) = \omega^*(F) = |F_\nu|^* = |\tilde{F}_\nu|^*$ ($F \in W^*(X)$), where $F_\nu \in Q^*(X)$ is the normal component of F, then we have

$$\sum |\tilde{F}_{n\nu}|^* = \sum \omega^*(\hat{F}_n) < \infty. \tag{1}$$

But $\{\tilde{F}_{n\nu}\}$ is a sequence in the Banach space $\tilde{Q}^*(X)$, and it follows from (1) and Proposition 2.3 that the series $\sum \tilde{F}_{n\nu}$ converges to $\tilde{F}\in\tilde{Q}^*(X)$. Since the mapping $r: \tilde{Q}^*(X) \longrightarrow \hat{W}^*(X)$, $\tilde{F} \longrightarrow \hat{F}$ is linear and continuous we must have

$$\sum \hat{F}_{n\nu}=\sum r(\tilde{F}_{n\nu})=r(\tilde{F})=\hat{F}. \tag{2}$$

But $\hat{F}=\hat{F}_\nu$ for $F\in W^*(X)$. Hence from (2) we obtain $\sum \hat{F}_n=\hat{F}$.

3. The B^*-numerical range.

DEFINITION 3.1. *Let $F\in W^*(X)$ and consider the continuous map $\phi_F : \Pi_0 \longrightarrow$ $\mathbf{C}$ given by*

$$\phi_F(x,f)=\frac{f(\langle F(x,f),x\rangle)}{\|x\|_X{}^2\|f\|}.$$

We define the B^*-numerical range $\Omega^*(F)$ of F as the set $\Omega^*(F)=\bigcap_{r>0}\overline{\phi_F(\Pi_r)}$. In other words, $\lambda\in\Omega^*(F)$ if and only if there exists a sequence $\{(x_n, f_n)\}$ in Π_0 such that $\|x_n\|_X\geqq n$ and

$$\frac{f_n(\langle F(x_n, f_n), x_n\rangle)}{\|x_n\|_X{}^2\|f_n\|} \longrightarrow \lambda \text{ as } n \longrightarrow \infty.$$

PROPOSITION 3.1. *If $F\in W^*(X)$, then $\Omega^*(F)$ is a nonempty compact connected subset of* $\mathbf{C}$.

Proof. Since $F\in W^*(X)$, then the sets $\overline{\phi_F(\Pi_r)}$ are bounded for all $r>0$ large enough. Now $\{\overline{\phi_F(\Pi_r)}\}$ is a nested family of compact nonempty sets, therefore by Cantor's theorem $\Omega^*(F)\neq\phi$ and is compact. Now from Proposition 2.1 we have that each $\overline{\phi_F(\Pi_r)}$ is a connected subset of $\mathbf{C}$. Thus $\Omega^*(F)$ being an intersection of a nested family of compact connected sets is connected as well.

The following properties of the B^*-numerical range are easy to check.

REMARK. If F, $G\in W^*(X)$ and $\lambda\in\mathbf{C}$, then

(a) $\Omega^*(F_\nu)=\Omega^*(F)$ and $\Omega^*(F_\tau)=\{0\}$.
(b) $\Omega^*(\lambda F)=\lambda\Omega^*(F)$.
(c) $\Omega^*(\lambda\pi+F) = \lambda + \Omega^*(F)$, where $\pi : X\times B^\# \longrightarrow X$ denotes the natural projection.
(d) $\Omega^*(F+G)\subseteqq\Omega^*(F)+\Omega^*(G)$.
(e) $\omega^*(F)=\max \{|\lambda| : \lambda\in\Omega^*(F)\}$. We call $\omega^*(F)$ the B^*-numerical radius of F.

PROPOSITION 3.2. *If $F,G\in W^*(X)$ and $\omega^*(F-G)=0$, then $\Omega^*(F)=\Omega^*(G)$.*

Proof. From the above Remark, we have $\Omega^*(F)=\Omega^*(F_\nu)$ and $\Omega^*(G)=\Omega^*(G_\nu)$. Also from Remark after Definition 2.5 we have $|F_\nu-G_\nu|^*=\omega^*(F-G)=0$. We shall show that $\Omega^*(F_\nu)=\Omega^*(G_\nu)$. Let $\lambda\in\Omega^*(F_\nu)$. Then there exists a sequence $\{(x_n, f_n)\}$ in Π_0 such that $\|x_n\|_X\geqq n$ and

$$\frac{f_n(\langle F_\nu(x_n,f_n), x_n\rangle)}{\|x_n\|_X{}^2\|f_n\|} \longrightarrow \lambda \text{ as } n \longrightarrow \infty.$$

Now

$$\frac{f_n(\langle G_\nu(x_n, f_n), x_n\rangle)}{\|x_n\|_X^2\|f_n\|} = \frac{f_n(\langle (G_\nu - F_\nu)(x_n, f_n), x_n\rangle)}{\|x_n\|_X^2\|f_n\|} + \frac{f_n(\langle F_\nu(x_n, f_n), x_n\rangle)}{\|x_n\|_X^2\|f_n\|} \tag{1}$$

But

$$\frac{|f_n(\langle (G_\nu - F_\nu)(x_n, f_n), x_n\rangle)|}{\|x_n\|_X^2\|f_n\|} \leqq \frac{\|(G_\nu - F_\nu)(x_n, f_n)\|_X}{\|x_n\|_X}$$

and $|G_\nu - F_\nu|^* = 0$ imply

$$\frac{f_n(\langle (G_\nu - F_\nu)(x_n, f_n), x_n\rangle)}{\|x_n\|_X^2\|f_n\|} \longrightarrow 0. \tag{2}$$

Hence from (1) and (2) we see that

$$\frac{f_n(\langle G_\nu(x_n, f_n), x_n\rangle)}{\|x_n\|_X^2\|f_n\|} \longrightarrow \lambda \text{ as } n \longrightarrow \infty.$$

Therefore $\Omega^*(F_\nu) \subseteqq \Omega^*(G_\nu)$. The inclusion $\Omega^*(G_\nu) \subseteqq \Omega^*(F_\nu)$ is proved in the same way.

PROPOSITION 3.3. *If* $F \in W^*(X)$, *then* (a) $\alpha^*(\mu\pi - F) \geqq dist\ (\mu, \Omega^*(F))$, $\mu \in \mathbf{C}$.
(b) $\Omega^*(F) = \{\lambda \in \mathbf{C}: \alpha^*(\lambda\pi - F) = 0\}$.

Proof. (a) We shall show a little more, namely; that for any $\mu \in \mathbf{C}$, there exists $\lambda \in \Omega^*(F)$ such that $\alpha^*(\mu\pi - F) = |\mu - \lambda|$. By definition of $\alpha^*(\mu\pi - F)$, there is a sequence $\{(x_n, f_n)\}$ in Π_0 such that $\|x_n\|_X \geqq n$ and

$$\frac{|f_n(\langle (\mu\pi - F)(x_n, f_n), x_n\rangle)|}{\|x_n\|_X^2\|f_n\|} \longrightarrow \alpha^*(\mu\pi - F) \tag{1}$$

Since $F \in W^*(X)$, without loss of generality we may assume that the sequence $\left\{\frac{f_n(\langle F(x_n, f_n), x_n\rangle)}{\|x_n\|_X^2\|f_n\|}\right\}$ is convergent to some $\lambda \in \Omega^*(F)$.
Thus from (1) we obtain $\alpha^*(\mu\pi - F) = |\mu - \lambda|$.
(b) Let $\Lambda = \{\lambda \in \mathbf{C}: \alpha^*(\lambda\pi - F) = 0\}$. Then from (a) we have $\Lambda \subseteqq \Omega^*(F)$. Now let $\lambda \in \Omega^*(F)$. Then there is a sequence $\{(x_n, f_n)\}$ in Π_0 such that $\|x_n\|_X \geqq n$ and $\frac{f_n(\langle F(x_n, f_n), x_n\rangle)}{\|x_n\|_X^2\|f_n\|} \longrightarrow \lambda.$

This in turn implies that $\frac{f_n(\langle (\lambda\pi - F)(x_n, f_n), x_n\rangle)}{\|x_n\|_X^2\|f_n\|} \longrightarrow 0$, and hence that $\alpha^*(\lambda\pi - F) = 0$. Therefore $\lambda \in \Lambda$ and $\Omega^*(F) \subseteqq \Lambda$.

PROPOSITION 3.4. *Let* $F, G \in W^*(X)$ *and* $\mu \in \mathbf{C}$. *Then*

(a) $0 \leqq \alpha^*(F) \leqq \omega^*(F)$.
(b) $\alpha^*(\mu F) = |\mu|\alpha^*(F)$.
(c) $\alpha^*(F+G) \leqq \alpha^*(F) + \omega^*(G)$
(d) $\alpha^*(F) - \omega^*(G) \leqq \alpha^*(F+G)$
(e) $|\alpha^*(F) - \alpha^*(G)| \leqq \omega^*(F-G)$. *So* α^* *is actually defined in* $\hat{W}^*(X)$.

(f) $\alpha^*(F) \leqq |\lambda|$ *if* $\lambda \in \Omega^*(F)$.

Proof. (a) an (b) follow from the definitions.

(c) $$\alpha^*(F+G) = \lim_{r\to\infty} \inf_{\Pi_r} \frac{|f(\langle (F+G)(x, f), x\rangle)|}{\|x\|_X{}^2\|f\|}$$
$$\leqq \lim_{r\to\infty} \inf_{\Pi_r} \frac{|f(\langle F(x, f), x\rangle)|}{\|x\|_X{}^2\|f\|} + \lim_{r\to\infty} \sup_{\Pi_r} \frac{|f(\langle G(x, f), x\rangle)|}{\|x\|_X{}^2\|f\|}$$
$$= \alpha^*(F) + \omega^*(G).$$

(d) It follows from (c).

(e) We have from (c) $\alpha^*(F) = \alpha^*(F-G+G) \leqq \omega^*(F-G) + \alpha^*(G)$. Hence $|\alpha^*(F) - \alpha^*(G)| \leqq \omega^*(F-G)$.

(f) From (d) and Proposition 3.3 (b) we have
$$\alpha^*(F) - |\lambda| \leqq \alpha^*(\lambda\pi - F) = 0, \quad \lambda \in \Omega^*(F).$$

PROPOSITION 3.5. *If* $F, G \in W^*(X)$, *then* $\gamma(\Omega^*(F), \Omega^*(G)) \leqq \omega^*(F-G)$. *Here* γ *is the Hausdorff metric in* $\Gamma(\mathbf{C})$, *which denotes the set of all non-void closed bounded subsets of* $(\mathbf{C}, d)$.

Proof. We have
$$\gamma(\Omega^*(F), \Omega^*(G)) = \max\{\sup\{\text{dist}(\lambda, \Omega^*(F)) : \lambda \in \Omega^*(G)\},$$
$$\sup\{\text{dist}(\lambda, \Omega^*(G)) : \lambda \in \Omega^*(F)\}\}. \qquad (1)$$
and from Proposition 3.3
$$\text{dist}(\lambda, \Omega^*(F)) \leqq \alpha^*(\lambda\pi - F), \quad \text{dist}(\lambda, \Omega^*(G)) \leqq \alpha^*(\lambda\pi - G). \qquad (2)$$
Also Proposition 3.3 and 3.4 (c) imply
$$\alpha^*(\lambda\pi - F) = \alpha^*((\lambda\pi - G) + (G-F)) \leqq \alpha^*(\lambda\pi - G) + \omega^*(G-F)$$
$$= \omega^*(F-G), \quad \lambda \in \Omega^*(G) \qquad (3)$$
and
$$\alpha^*(\lambda\pi - G) = \alpha^*((\lambda\pi - F) + (F-G)) \leqq \alpha^*(\lambda\pi - F) + \omega^*(F-G)$$
$$= \omega^*(F-G), \quad \lambda \in \Omega^*(F). \qquad (4)$$
From (1)−(4) we obtain $\gamma(\Omega^*(F), \Omega^*(G)) \leqq \omega^*(F-G)$.

4. The numerical range and the B^*-asymptotic spectrum

DEFINITION 4.1. *Let* $X_0 = X - \{0\}$. *A continuous map* $P : X_0 \longrightarrow X$ *is called B-numerically bounded if the map* $F : \Pi_0 \longrightarrow X$ *given by* $F(x, f) = P(x)$ *is* B^*-*numerically bounded.*

In this case the numbers $\omega^*(F)$, $\alpha^*(F)$ and the B^*-numerical range $\Omega^*(F)$ are denoted by $\omega(P)$, $\alpha(P)$ and $\Omega(P)$ respectively.

We denote by $W(X)$ the vector space of all B-numerically bounded maps on X_0. Notice that $W(X)$ can be considered, in a natural way, as a vector subspace of $W^*(X)$, and that ω is a seminorm on $W(X)$. Obviously one has $B(X) \subset Q(X) \subset W(X)$ and $\omega(P) \leqq |P| \leqq \|P\|$.

PROPOSITION 4.1. *If* $T \in L(X)$, *then*

(a) $\Omega(T) = \overline{W_B(T)}$.

(b) $\omega(T)=\omega_B(T)$, *where* $\omega_B(T)$ *denotes the B-spatial numerical radius of* T[7].

Proof. Obvious.

DEFINITION 4.2. *For any* $F\in Q^*(X)$, *we define*

$$d^*(F)=\lim_{r\to\infty}\inf_{\Pi r}\frac{\|F(x,f)\|_X}{\|x\|_X},$$

and the B^*-asymptotic spectrum $\Sigma^*(F)$ of F, as the set $\Sigma^*(F)=\{\lambda\in\mathbf{C}: d^*(\lambda\pi-F)=0\}$ where π denotes the natural projection of $X\times B^\#$ onto X.
It is easy to show the following properties; If $F,G\in Q^*(X)$ and $\mu\in\mathbf{C}$, then

(a) $0\leqq d^*(F)\leqq|F|^*$.
(b) $d^*(\mu F)=|\mu|d^*(F)$
(c) $d^*(F+G)\leqq d^*(F)+|G|^*$.
(d) $d^*(F)-|G|^*\leqq d^*(F+G)$
(e) $|d^*(F)-d^*(G)|\leq|F-G|^*$.
(f) $d^*(F)\leqq|\lambda|,\ \lambda\in\Sigma^*(F)$.

PROPOSITION 4.2. *If* F, $G\in Q^*(X)$ *and* $\mu\in\mathbf{C}$, *then*
(a) $\Sigma^*(F)\subseteqq\Omega^*(F)$.
(b) *If* $|F-G|^*=0$, *then* $\Sigma^*(F)=\Sigma^*(G)$.
(c) $r^*(F)\leqq|F|^*$, *where* $r^*(F)=\sup\{|\lambda|:\lambda\in\Sigma^*(F)\}$ *is the* B^**-asymptotic spectral radius of* F.
(d) $\Sigma^*(F)$ *is compact.*
(e) $\Sigma^*(\mu F)=\mu\,\Sigma^*(F)$.
(f) $\Sigma^*(\mu\pi+F)=\mu+\Sigma^*(F)$.

Proof. (a) It follows from the obvious inequality $\alpha^*(F)\leqq d^*(F)$ and Proposition 3.3.
(b) Immediate from the previous remark (e).
(c) Let $\lambda\in\Sigma^*(F)$. By the previous remark (d) we have

$$|\lambda|-|F|^*\leqq d^*(\lambda\pi-F)=0.$$

(d) By the previous remark (e), the mapping $\lambda\longrightarrow d^*(\lambda\pi-F)$ is continuous and hence $\Sigma^*(F)$ is closed. By (c) it is bounded and hence compact.

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Pusan National University
Pusan 607, Korea
and
Korea Naval Academy
Jinhae 602, Korea