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A ROLE OF *K*-GROUPS IN THE CLASSIFICATION OF *AF* ALGEBRAS

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1. Introduction

In recent years various functors on C^*-algebras have been introduced to the realm of C^*-algebras. Among them are the following. With techniques usually used in algebraic topology Brown-Douglas-Fillmore [1] created the extension group Ext (A) and they used it to solve various problems originated from operator theory. Then Pimsner-Popa-Voiculescu generalized these and they obtained more complete picture of Ext $(Y;A)$. In addition there is the variant of K-theory for Banach algebras [7].

The K_0-group of an AF algebras has been studied more extensively. These groups have the structure of the ordered "dimension group". This dimension group classifies AF algebras completely [4].

In this note we use the K_0-group of the rather special class of AF algebras, the so-called UHF algebras to the classification problem. Our result says that for two UHF algebras A, and B, $K_0(A)$ and $K_0(B)$ are isomorphic if and only if there exist natural numbers n and m such that $M_n(A)$ and $M_m(B)$ are $*$-isomorphic. Our result is stronger than Elliot's earlier result (see [3], [4]). And we mention that our technique is more explicit and we do not use the order structure of K_0-group. (i.e., dimension group)

2. Generalities

Let A be a C^*-algebra with unit. If there exists a sequence of finite-dimensional C^*-subalgebras A_n of A with the same unit such that

(1) A_n is a C^*-subalgebra of A_{n+1} with the same unit
(2) $A=\overline{\cup A_n}$

then we call A *approximately finite-dimensional* (AF) algebra. In particular, if all A_n's are simple finite-dimensional C^*-algebras (i.e., one $*$-isomorphic to a full matrix algebra), then such an AF algebra is called *uniformly hyperfinite* (UHF). Let A be a UHF algebra with $A=\overline{\cup A_n}$. Let A_k be $*$-isomorphic to $n_k \times n_k$ matrix algebras over the complex numbers $\mathbf{C}$ (we will denote it by $M_{n_k}(\mathbf{C})$.

Let θ_k be the unital inclusion map from $M_{n_k}(\mathbf{C})$ into $M_{n_{k+1}}(\mathbf{C})$ via the inclusion of A_n into A_{n+1}. Let $p_1, \ldots, p_{n_k}$ be mutually equivalent orthogonal projections in $M_{n_k}(\mathbf{C})$ such that $p_1 + \ldots + p_{n_k} = 1$. Then $\theta_k(p_1), \ldots, \theta_k(p_{n_k})$ are mutually equivalent projections, in $M_{n_{k+1}}(\mathbf{C})$. Let m denote the integral dimension of the projection $\theta_k(p_1)$ in $M_{n_{k+1}}(\mathbf{C})$. Then $n_k \times m = n_{k+1}$. Hence n_k divides n_{k+1} for all k. Thus A can be viewed as $A = \overline{\cup_k M_{n_k}(\mathbf{C})}$, where n_k divides n_{k+1} for all k.

We briefly describe the construction of K_0-group of a general C^*-algebra A. Details can be found in [7]. Let $M_n(A)$ denote the set of $n \times n$ matrices of entries from A. Then it is easy to see that $M_n(A)$ is actually a C^*-algebra with the usual matrix addition, matrix multiplication and with the natural C^*-norm. Let $\mathrm{Proj}_n(A)$ be the set of all projections in $M_n(A)$. Then we have natural injections

$$\mathrm{Proj}_n(A) \to \mathrm{Proj}_{n+1}(A)$$

defined by $e \to e \oplus 0$. We define an equivalence relation on $\mathrm{Proj}_n(A)$ by letting $e \simeq f$ if there is a unitary u in $M_n(A)$ such that $u^*eu = f$. Then it is easy to see that the inclusions respect this equivalence relation on $M_n(A)$. We denote $D_n(A)$ the set of equivalence classes. Then we have a system of sets

$$D_n(A) \to D_{n+1}(A)$$

for any e, f in $D_n(A)$ we define

$$e + f = (e \oplus 0) + (0 \oplus f)$$

in $D_n(A)$.

Then it is routine to check that the inductive limit $D(A)$ of $D_n(A)$ with the addition defined as above gives us a semi-group $D(A)$. Finally the Grothendieck group for $D(A)$ is called the *K_0-group* $K_0(A)$. It is easy to see that $K_0(M_n(\mathbf{C})) \simeq \mathbf{Z}$, the integer group, and that the inclusion $i : M_n(\mathbf{C}) \to M_{np}(\mathbf{C})$ induces $i_* : K_0(M_n(\mathbf{C})) \to K_0(M_{np}(\mathbf{C}))$ and $i_*(1) = p$. It is not hard to see that K_0 is a covariant functor from the category of C^*-algebras to the category of abelian groups. We close this section with the following lemma.

LEMMA 1. *Suppose that A is a C^*-algebra and that $A_1, A_2, \ldots$ is an increasing sequence of C^*-subalgebras with $A = \overline{\cup A_n}$. Then we have* $K_0(A) \simeq \varinjlim K_0(A_n)$

Proof. See [6] for example.

3. Classifications

Throughout this section K will denote the compact ideal of the algebra of all bounded linear operators on a separable infinite-dimensional Hilbert space H. For a unital C^*-algebra A, $K \otimes A$ will denote the C^*-algebra of tensor product of K and A.

THEOREM. *Let A and B be UHF algebras with $A = \overline{\cup M_{n_k}(\mathbf{C})}$ and $B = \overline{\cup M_{m_k}(\mathbf{C})}$. Then the following are equivalent:*

(i) $K \otimes A \simeq K \otimes B$

(ii) $K_0(A) \simeq K_0(B)$

(iii) *There exist two natural numbers n' and n'' such that*

$$M_{n'}(A) \simeq M_{n''}(B).$$

Proof. (i) → (ii). If e and f are projections in $K \otimes A$, we write $e \sim f$ if there is a partial isometry u in $K \otimes A$ such that $u^*u = e$ and $uu^* = f$. Then the relation "$\sim$" is an equivalence relation on $\mathrm{Proj}(K \otimes A)$ and we denote by $[e]$ the equivalence class. Then

$$S = \{[e] \mid e \in \mathrm{Pro}(K \otimes A)\}$$

is an abelian semigroup with the addition $[e]+[f]=[e'+f']$, where $e \sim e'$, $f \sim f'$, and $e'f'=0$ (it is always possible to find such e' and f' in $\mathrm{Proj}(K \otimes A)$). Then it is well-known that the Grothendieck group of this semi-group S is the same as $K_0(A)$ (see [2], [3]). Now since $K \otimes A \simeq K \otimes B$, respective semi-groups are isomorphic, so are their K_0-groups. Note that this arguement applies to any unital C^*-algebra.

(ii)→(iii) Let $a_k = n_{k+1}/n_k$ and $b_k = m_{k+1}/m_k$. Let $m(a_k)(n) = a_k \times n$ for all $n \in N$. By Lemma 1 and paragraphs preceding it, we see that $K_0(A) \simeq \varinjlim (Z_k, m(a_k))$ and $K_0(B) \simeq \varinjlim (Z_k, m(b_k))$, where $Z_k = \mathbf{Z}$ for all k. Let θ be the given isomorphism: $\varinjlim (Z_k, m(a_k)) \to \varinjlim (Z_k, m(b_k))$. Then there exists a smallest integer k_1 such that $\theta((1, 0, \dots)) = (0, \dots, d_1, 0, \dots)$, where d_1 is nonzero integer in the k_1-position. Then there is a smallest integer p_1 such that $\theta^{-1}((0, \dots, 0, 1, 0, \dots))$ is of the form $(0, \dots, 0, e_1, 0, \dots)$ where e_1 is in the p_1-th position. Continuing this process we get a direct system of abelian groups

$$Z_{p_0} \to Z_{k_1} \to Z_{p_1} \to Z_{k_2} \to \dots$$

where the map $Z_{p_{i-1}} \to Z_{k_i}$ is the multiplication by d_i and the map $Z_{k_i} \to Z_{p_i}$ is the multiplication by e_i and $p_0 = 1$. Then we have

$$a_1 \times a_2 \times \cdots \times a_{n-1} = d_1 \times b_{k_1} \times \cdots \times b_{k_{n-1}} \times e_n \quad \text{for } n > 1,$$

and

$$a_1 \times a_2 \times \cdots \times a_{n-1} \times d_n = d_1 \times b_{k_1} \times \cdots \times b_{k_{n-1}} \quad \text{for } n > 1.$$

Let C be the UHF algebra corresponding to the sequence $\{b_{k_1}, b_{k_2}, \dots, b_{k_n}\}_{n=1}^{\infty}$. Then by Glimm's theorem [5], $A \simeq M_{d_1}(C)$ and $B \simeq M_d(C)$, where $d_1 = b_1 \times \dots \times b_{k_1 - 1}$. Let $d = n'$ and $d_1 = n''$. Then $M_{n'}(A) \simeq M_{n''}(B)$.

(iii)→(i). Since $K \simeq M_{n'}(K)$ and $K \simeq M_{n''}(K)$, (iii) implies (i).

REMARK. It would be of some interest to prove that (iii) implies (i) directly.

References

1. L. G. Brown, R. G. Douglas, and P. A. Fillmore, *Extensions of C^*-algebras*

and K-homology, Ann. of Math. (2) **105** (1977), 265-324.
2. J. Cuntz, *K-theory for certain C^*-algebras* I, Ann. of Math. (2) **113** (1981), 181-197.
3. E. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conf. Ser. Math. Amer. Math. Soc. 1981.
4. G. Elliott, *On the classification of inductive limits of sequence of, semi-simple finite dimensional algebras*, J. Alg. **38** (1976), 29-44.
5. J. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. **95** (1960), 318-340.
6. D. Handelman, *K_0 of von Neumann algebras and AF C^*-algebras*, Quart. J. Math. Oxford (2), **29** (1978), 427-441.
7. J.L. Taylor, *Banach algebras and topology, algebras and analysis*, ed. by J.H. Williamson, Academic Press, N.Y., 1975.

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