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EVALUATION OF SOME CONDITIONAL WIENER INTEGRALS

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1. Introduction

J. Yeh has recently introduced the concept of conditional Wiener integrals which are meant specifically the conditional expectation $E^w(Z|X)$ of a real or complex valued Wiener integrable functional Z conditioned by the Wiener measurable functional X on the Wiener measure space (A precise definition of the conditional Wiener integral and a brief discussion of the Wiener measure space are given in Section 2). In [3] and [4] he derived some inversion formulae for conditional Wiener integrals and evaluated some conditional Wiener integrals $E^w(Z|X)$ conditioned by $X(x)=x(t)$ for a fixed $t>0$ and x in Wiener space. Thus $E^w(Z|X)$ is a real or complex valued function on $\mathbf{R}^1$.

In this paper we shall be concerned with the random vector X given by $X(x)=(x(s_1),\ldots,x(s_n))$ for every x in Wiener space where $0=s_0<s_1<\ldots<s_n=t$. In Section 3 we will evaluate some conditional Wiener integrals $E^w(Z|X)$ which are real or complex valued functions on the n-dimensional Euclidean space $\mathbf{R}^n$. Thus we extend Yeh's results [4] for the random variable X given by $X(x)=x(t)$ to the random vector X given by $X(x)=(x(s_1),\ldots,x(s_n))$.

2. Preliminaries

For a fixed $t>0$ let $T=[0,t]$. Let $C(T)$ be the collection of all $\mathbf{R}$-valued continuous funtions x defined on T such that $x(0)=0$. The space $C(T)$ with the uniform topology is called the Wiener space. Consider the Wiener measure space $(C(T),\ \mathfrak{W}^*,\ m_w)$ where $\mathfrak{W}$ is the algebra of subsets I of $C(T)$ of the type

$$I=\{x\in C(T)\ :\ (x(s_1),\ \ldots,\ x(s_n))\in B\} \tag{2.1}$$

where n is an arbitrary natrural number, $0=s_0<s_1<\ldots<s_n\leqq t$, and B is an arbitrary member of the σ-algebra $\mathcal{B}(\mathbf{R}^n)$ of the Borel sets in $\mathbf{R}^n$; m_w is a probability measure on the algebra $\mathfrak{W}$ defined for I as in (2.1) by

$$m_w(I)=\Big\{(2\pi)^n\prod_{i=1}^n(s_i-s_{i-1})\Big\}^{-\frac{1}{2}} \int_B\exp\Big\{-(1/2)\sum_{i=1}^n(\eta_i-\eta_{i-1})^2/(s_i-s_{i-1})\Big\}\,dm_L(\eta) \tag{2.2}$$

where $\eta=(\eta_1,\ldots,\eta_n)\in\mathbf{R}^n$, $\eta_0=0$ and m_L is the Lebesgue measure on $(\mathbf{R}^n,\ \mathcal{B}(R^n))$; $\mathfrak{W}^*$ is the σ-algebra of Carathéodory measurable subsets of $C(T)$ with respect to the outer measure derived from the probability measure m_w on the algebra $\mathfrak{W}$

which in particular contains the σ-algebra $\sigma(\mathfrak{W})$ generated by $\mathfrak{W}$.

A real valued functional F on $C(T)$ is said to be Wiener measurable if it is a measurable transformation of $(C(T), \mathfrak{W}^*)$ into $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1))$. For a Wiener measurable functional F we write

$$(2.3)\qquad E^w(F) \quad \text{for} \quad \int_{C(T)} F(x)\, dm_w(x)$$

whenever the integral, i.e. the Wiener integral, exists. A real valued Wiener measurable functional F on $C(T)$ is said to be Wiener integrable or m_w-integrable if the Wiener integral of F exists and is finite.

The following theorem shows the relation between the Wiener integal and the Lebesgue integral. We need it in the following section. We will state it without proof [5].

THEOREM 2.1. *If F is a real or complex valued functional on $C(T)$ of the type*

$$(2.4)\qquad F(x)=f(x(s_1),\dots,x(s_n)) \quad for\ x\in C(T)$$

where f is a real or complex valued Lebesgue measurable function on $\mathbf{R}^n$ and $0<s_1<\dots<s_n\leq t$ then F is Wiener measurable and

$$(2.5)\qquad E^w(F) \overset{*}{=} \left\{(2\pi)^n \prod_{i=1}^{n} (s_i-s_{i-1})\right\}^{-\frac{1}{2}} \int_{\mathbf{R}^n} f(\eta) \exp\left\{-(1/2)\sum_{i=1}^{n} (\eta_i-\eta_{i-1})^2/(s_i-s_{i-1})\right\} dm_L(\eta)$$

where throghout this paper the notation "$\overset{*}{=}$" means that the existence of one side in (2.5) implies that of the other as well as the equality of the two.

Let X be a $\mathbf{R}^n$-valued Wiener measurable function on the Wiener measure space $(C(T), \mathfrak{W}^*, m_w)$. The set function P_X on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ defined by $P_X(B)=m_w(X^{-1}(B))$, $B\in\mathcal{B}(\mathbf{R}^n)$, is a probability measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ and is called the n-dimensional probability distribution determined by the random vector X.

Let X and Z be the $\mathbf{R}^n$-valued and the real valued Wiener measurable functionals on the Wiener measure space $(C(T), \mathfrak{W}^*, m_w)$ respectively with $E^w(|Z|)<\infty$. The equivalence class of $\mathcal{B}(\mathbf{R}^n)$-measurable and P_X-integrable functions f on $\mathbf{R}^n$ satisfying

$$(2.6)\qquad \int_{X^{-1}(B)} Z(x)\, dm_w(x) = \int_B f(u)\, dP_X(u)$$

for every B in $\mathcal{B}(\mathbf{R}^n)$ is called the conditional Wiener integral of Z given by X and is denoted by $E^w(Z|X)$, the equivalence relation being that of a.e. equality with respect to P_X.

From the Radon-Nikodym Theorem follows that such a function f always exists and is determined uniquely up to a null set in $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), P_X)$. We shall use $E^w(Z|X)$ to mean either the class of all such functions f or a particular version in it depending on the context. Thus

$$(2.7)\qquad \int_{X^{-1}(B)} Z(X)\ dm_w(x) = \int_B E^w(Z|X)(u)\,dP_X(u)$$

for every B in $\mathcal{B}(\mathbf{R}^n)$.

3. The evaluation of some conditional Wiener integrals

In this section we introduce the inversion formula for conditional Wiener integrals and then we will evaluate some conditional Wiener integrals $E^w(Z|X)$ of a real or complex valued Wiener integrable functional Z conditioned by $X(x) = (x(s_1), \ldots, x(s_n))$ for every x in $C(T)$ where $0 = s_0 < s_1 < \ldots < s_n = t$.

PROPOSITION 3.1. *Let X and Z be the measurable transformation of $(C(T), \mathfrak{W}^*)$ into $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ and $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1))$, respectively, with $E^w(|Z|) < \infty$. Let g be a measurable transformation of $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ into $(\mathbf{R}^1, \mathcal{B}(R^1))$. Then*

$$(3.1)\quad E^w((g\circ X)Z) \stackrel{*}{=} \int_{R^n} g(\eta_1, \ldots, \eta_n) E^w(X|X)(\eta_1, \ldots, \eta_n)\, dP_X(\eta_1, \ldots, \eta_n).$$

Proof. This proposition is a particular case of Proposition 3 in [3], that is, we adopted our probability space $(C(T), \mathfrak{W}^*, m_w)$ and the measurable space $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

PROPOSITION 3.2. *Let X and Z be as in Proposition 3.1. Assume that $P_X \ll m_L$ on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$. For $(\eta_1, \ldots, \eta_n) \in \mathbf{R}^n$ and positive numbers $r_1, r_2, \ldots,$ and r_n, let $J^{(r_1, \ldots, r_n)}_{(\eta_1, \ldots, \eta_n)}$ be a function on $\mathbf{R}^n$ defined by*

$$J^{(r_1, \ldots, r_n)}_{(\eta_1, \ldots, \eta_n)}(s_1, \ldots, s_n) = \begin{cases} (2^n r_1 r_2 \ldots r_n)^{-1} & \text{for } (s_1, \ldots, s_n) \in \mathop{X}\limits_{i=1}^{n} [\eta_i - r_i, \eta_i + r_i] \\ 0 & \text{for } (s_1, \ldots, s_n) \notin \mathop{X}\limits_{i=1}^{n} [\eta_i - r_i, \eta_i + r_i]. \end{cases}$$

Then there exists a version of $E^w(Z|X)\dfrac{dP_X}{dm_L}$ such that

$$E^w(Z|X)(\eta_1, \ldots, \eta_n)\frac{dP_X}{dm_L}(\eta_1, \ldots, \eta_n) = \lim_{(r_1, \ldots, r_n)\to(0, \ldots, 0)} E^w\left[\left(J^{(r_1, \ldots, r_n)}_{(\eta_1, \ldots, \eta_n)} \circ X\right) Z\right]$$

for $(\eta_1, \ldots, \eta_n) \in \mathbf{R}^n$.

Proof. Substituting $J^{(r_1, \ldots, r_n)}_{(\eta_1, \ldots, \eta_n)}$ in the place of g in Proposition 3.1, we have

$$\lim_{(r_1, \ldots, r_n)\to(0, \ldots, 0)} E^w\left(\left(J^{(r_1, \ldots, r_n)}_{(\eta_1, \ldots, \eta_n)} \circ X\right) Z\right)$$
$$= \lim_{(r_1, \ldots, r_n)\to(0, \ldots, 0)} \int_{\mathbf{R}^n} J^{(r_1, \ldots, r_n)}_{(\eta_1, \ldots, \eta_n)}(s_1, \ldots, s_n) E^w(Z|X)(s_1, \ldots, s_n)$$

$$\frac{dP_X}{dm_L}(s_1, \ldots, s_n)\ dm_L\ (s_1, \ldots, s_n).$$

Let $f=E^w(Z|X)\dfrac{dP_X}{dm_L}$. Then f is m_L-integrable since $P_X \ll m_L$ and $E^w(Z|X)$ is P_X-integrable. It is well known that if f is m_L-integrable on $\mathbf{R}^n$, then

$$\lim_{(r_1, \ldots, r_n)\to(0, \ldots, 0)} (2^n r_1 \ldots r_n)^{-1}\int_{\overset{n}{\underset{i=1}{X}}[\eta_i - r_i, \eta_i + r_i]} f(u)\ dm_L(u) = f(\eta_1, \ldots, \eta_n) \tag{3.2}$$

for a.e. $(\eta_1, \ldots, \eta_n)$ in $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), m_L)$ [1; p.78]. From (3.2) follows our Proposition 3.2.

For $x\in C(T)$ consider the average value of x over the time interval T, i.e. $(1/t)\int_0^t x(s)\ ds$. Now we evaluate the conditional Wiener integral for Wiener integrable functionals conditioned by $X(x)=(x(s_1), \ldots, x(s_n))$ where $0=s_0<s_1<\ldots<s_n=t$.

THEOREM 3.1. *For $x\in C(T)$, let Z be the average value of x over the time interval T, i.e. $Z(x)=(1/t)\int_0^t x(s)\ ds$, and $X(x)=(x(s_1), \ldots, x(s_n))$ where $0=s_0<s_1<\ldots<s_n=t$. Then the conditional Wiener integral of Z given X is*

$$E^w(Z|X)(\eta_1, \ldots, \eta_n)=(1/2t)\sum_{i=1}^n (s_i-s_{i-1})(\eta_{i-1}+\eta_i)$$

where $x(s_i)=\eta_i$ for $i=0, 1, \ldots, n$.

Proof. Since $|Z(x)|\leqq(1/t)\int_0^t |x(u)|\ du$ and

$$E^w(|x(u)|)=(2\pi u)^{-\frac{1}{2}}\int_{\mathbf{R}^1}|w|\exp(-w^2/2u)\ dm_L(w)=(2u/\pi)^{\frac{1}{2}},$$

we have

$$E^w((1/t)\int_0^t |x(u)|du)=(1/t)\int_0^t E^w(|x(u)|)\ du=(2\sqrt{2}/3)\ (t/\pi)^{\frac{1}{2}}.$$

Therefore $E^w(|Z|)$ is finite and so $E^w(Z|X)$ exists.
According to Proposition 3.2, a version of $E^w(Z|X)\dfrac{dP_X}{dm_L}$ is given by

$$E^w(Z|X)(\eta_1, \ldots, \eta_n)\frac{dP_X}{dm_L}(\eta_1, \ldots, \eta_n)=\lim_{(r_1, \ldots, r_n)\to(0, \ldots, 0)} E^w((J\binom{r_1, \ldots, r_n}{\eta_1, \ldots, \eta_n}\circ X)Z) \tag{3.3}$$

where $(\eta_1, \ldots, \eta_n)\in\mathbf{R}^n$. With our Z and X, we have

$$E^w((J\binom{r_1, \ldots, r_n}{\eta_1, \ldots, \eta_n}\circ X)Z)=\sum_{i=1}^n I_i \tag{3.4}$$

were

$$I_i=E^w[(J\binom{r_1, \ldots, r_n}{\eta_1, \ldots, \eta_n}(x(s_1), \ldots, x(s_n))(1/t)\int_{s_{i-1}}^{s_i} x(u)\ du]$$

for $i=1, 2, \ldots, n$.
Here $X((s_1, \ldots, s_n), x)=(x(s_1), \ldots, x(s_n))$ for $((s_1, \ldots, s_n), x)\in T^n\times C(T)$

is Lebesgue $\times$ Wiener measurable since it is continuous on the product space $T^n \times C(T)$. To apply the Fubini's Theorem, observe that

$$\left| J\begin{smallmatrix}(r_1, \ldots, r_n)\\(\eta_1, \ldots, \eta_n)\end{smallmatrix}(x(s_1), \ldots, x(s_n))x(u)\right| \leqq (1/2^n r_1 \ldots r_n)\,|x(u)|$$

for $(u, x) \in T \times C(T)$ and by (2.5)

$$\int_T E^w((1/2^n r_1 \ldots r_n)\,|x(u)|)\ du$$
$$= (1/2^n r_1 \ldots r_n)\int_T\left\{(2\pi u)^{-\frac{1}{2}}\int_{R^1}|w|\exp(-w^2/2u)\ dm_L(w)\right\}dm_L(u)$$
$$= (1/2^n r_1 \ldots r_n)\ (2t)^{\frac{3}{2}}\ /\ (3\pi^{\frac{1}{2}}) < \infty.$$

hus by the Fubini's Theorem and (2.5), we have

$$I_i = (1/t2^n r_1 \ldots r_n)\int_{(s_i-1, s_i)}\left\{(2\pi)^{n+1}(u-s_{i-1})(s_i-u)\prod_{\substack{j=1\\j\neq 1}}^{n}(s_j-s_{j-1})\right\}^{-\frac{1}{2}}$$

$$\text{(3.5)}\qquad \left\{\int_{\substack{X\\ i=1}}^{\times [\eta_i - r_i, \eta_i + r_i]}\int_{R^1}\beta\exp\left\{-\frac{(\beta-\alpha_{i-1})^2}{2(u-s_{i-1})}-\frac{(\alpha_i-\beta)^2}{2(s_i-u)}\right\}\right.$$

$$\left.\exp\left\{-\frac{1}{2}\sum_{\substack{j=1\\j\neq i}}^{n}\frac{(\alpha_j-\alpha_{j-1})^2}{s_j-s_{j-1}}\right\}\ d\beta\, d\alpha_1 \ldots d\alpha_n\right\}\ dm_L(u)$$

for $i = 1, 2, \ldots, n$. Here

$$\text{(3.6)}\qquad \int_{-\infty}^{\infty}\beta\exp\left\{-\frac{(\beta-\alpha_{i-1})^2}{2(u-s_{i-1})}-\frac{(\alpha_i-\beta)^2}{2(s_i-u)}\right\}d\beta$$
$$= \frac{(s_i-u)\alpha_{i-1}+(u-s_{i-1})\alpha_i}{s_i-s_{i-1}}\sqrt{\frac{2\pi(u-s_{i-1})(s_i-u)}{s_i-s_{i-1}}}\ \exp\left\{-\frac{(\alpha_i-\alpha_{i-1})^2}{2(s_i-s_{i-1})}\right\}.$$

The equality in (3.6) follows from the formula;

$$\text{(3.7)}\qquad \frac{1}{\sqrt{2\pi v}}\int_{-\infty}^{\infty}y^p\exp\{-y^2/2v\}\,dm_L(y) = 1, 0,\ v \text{ for } p = 0, 1, 2$$

respectively, with $v > 0$. Substituting (3.6) for the right-hand side of (3.5), we have

$$\text{(3.8)}\qquad I_i = \frac{s_i - s_{i-1}}{2^{n+1}r_1 \ldots r_n t}\int_{\substack{n\\ \times\\ i=1}[\eta_i - r_i, \eta_i + r_i]}(\alpha_{i-1}+\alpha_i)\left\{(2\pi)^n\prod_{i=1}^{n}(s_i-s_{i-1})^{-\frac{1}{2}}\right.$$
$$\exp\left\{(-1/2)\sum_{i=1}^{n}(\alpha_i-\alpha_{i-1})^2/(s_i-s_{i-1})\right\}d\alpha_1, \ldots d\alpha_n$$

for $i = 1, 2, \ldots, n$. Substituting (3.8) for the right-hand side of (3.4) and by (3.3) and (3.2), we have

$$E^w(Z|X)(\eta_1, \ldots, \eta_n)\frac{dP_X}{dm_L}(\eta_1, \ldots, \eta_n)$$
$$= (1/2t)\sum_{i=1}^{n}(s_i-s_{i-1})(\eta_i+\eta_{i-1})\left\{(2\pi)^n\prod_{i=1}^{n}(s_i-s_{i-1})\right\}^{-\frac{1}{2}}$$
$$\exp\left\{(-1/2)\sum_{i=1}^{n}(\eta_i-\eta_{i-1})^2/(s_i-s_{i-1})\right\}.$$

Therefore we obtain

$$E^w(Z|X)(\eta_1, \ldots, \eta_n) = (1/2t) \sum_{i=1}^{n} (s_i - s_{i-1})(\eta_{i-1} + \eta_i)$$

since

$$\frac{dP_X}{dm_L}(\eta_1, \ldots, \eta_n) = \left\{(2\pi)^n \prod_{i=1}^{n} (s_i - s_{i-1})\right\}^{-\frac{1}{2}} \exp\left\{(-1/2) \sum_{i=1}^{n} (\eta_i - \eta_{i-1})^2 / (s_i - s_{i-1})\right\}. \tag{3.9}$$

It is of interest to note that from (2.7), (3.7), and by Theorem 3.1

$$\begin{aligned} E^w(Z) &= \int_{X^{-1}(\mathbf{R}^n)} Z(x)\, dm_w(x) = \int_{\mathbf{R}^n} E^w(Z|X)(x)\, dP_X(x) \\ &= \left\{(2\pi)^n \prod_{i=1}^{n} (s_i - s_{i-1})\right\}^{-\frac{1}{2}} \int_{\mathbf{R}^n} \left\{\sum_{i=1}^{n} (s_i - s_{i-1})(\eta_{i-1} + \eta_i)/(2t)\right\} \\ &\quad \exp\left\{(-1/2) \sum_{i=1}^{n} (\eta_i - \eta_{i-1})^2 / (s_i - s_{i-1})\right\} d\eta_1 \ldots d\eta_n = 0. \end{aligned}$$

This is the same result as a direct computation of $E^w(Z)$.

REMARK. Let Z be given as in Theorem 3.1. For $x \in C(T)$, let $X(x) = x(t) = \xi \in \mathbf{R}^1$, then $E^w(Z|X)(\xi) = \xi/2$. Thus the result that J. Yeh obtained in [4, Example 1, p. 629] is a special case of Theorem 3.1 with $n=1$. And also let $X(x) = (x(s), x(t)) = (\eta, \xi) \in \mathbf{R}^2$, then $E^w(Z|X)(\eta, \xi) = \eta/2 + (t-s)\xi/(2t)$. It is clear that $\lim_{s \to 0} E^w(Z|X)(\eta, \xi) = \lim_{s \to t} E^w(Z|X)(\eta, \xi) = \xi/2$.

THEOREM 3.2. *For* $x \in C(T)$, *let* $X(x) = (x(s_1), \ldots, x(s_n))$ *and* $Z(x) = \sum_{i=1}^{n} (1/(s_i - s_{i-1})) \int_{(s_i-1, s_i)} x(u)\, dm_L(u)$ *where* $0 = s_0 < s_1 < \ldots < s_n = t$. *Then the conditional Wiener integral of* Z *given* X *is*

$$E^w(Z|X)(\eta_1, \ldots, \eta_n) = \sum_{i=0}^{n-1} \eta_i + \eta_n/2$$

where $x(s_i) = \eta_i$ *for* $i = 0, 1, \ldots, n$.

Proof. It is obvious that $E^w(Z|X)$ exists. By Proposition 3.2 and our Z and X, a version of $E^w(Z|X)\frac{dP_X}{dm_L}$ is given by

$$\begin{aligned} &E^w(Z|X)(\eta_1, \ldots, \eta_n) \frac{dP_X}{dm_L}(\eta_1, \ldots, \eta_n) \\ &= \lim_{(r_1, \cdots, r_n) \to (0, \cdots, 0)} E^w[J\binom{r_1, \ldots, r_n}{\eta_1, \ldots, \eta_n}(x(s_1, \ldots, x(s_n)) \sum_{i=1}^{n} (1/(s_i - s_{i-1})) \\ &\quad \int_{(s_i-1, s_i)} x(u)\, dm_L(u)]. \end{aligned} \tag{3.10}$$

Let

$$\begin{aligned} I_i^* = E^w((J\binom{r_1, \ldots, r_n}{\eta_1, \ldots, \eta_n}(x(s_1), \ldots, x(s_n))(1/(s_i - s_{i-1})) \\ \int_{(s_i-1, s_i)} x(u)\, dm_L(u)] \end{aligned}$$

for $i=1,2,\ldots,n$. Then by (3.8) and (3.2), we have

(3.11) $$\lim_{(r_1,\cdots,r_n)\to(0,\cdots,0)} I^*$$
$$=((\eta_{i-1}+\eta_i)/2)\left\{(2\pi)^n\prod_{i=1}^n(s_i-s_{i-1})\right\}^{-\frac{1}{2}}\exp\left\{(-1/2)\sum_{i=1}^n(\eta_i-\eta_{i-1})^2/(s_i-s_{i-1})\right\}$$

for $i=1, 2, \ldots, n$. Substituting (3.11) for the right-hand side of (3.10), we obtain

$$E^w(Z|X)(\eta_1,\ldots,\eta_n)\frac{dP_X}{dm_L}(\eta_1,\ldots,\eta_n)$$
$$=\sum_{i=1}^n\left\{(\eta_{i-1}+\eta_i)/2\right\}\left\{(2\pi)^n\prod_{i=1}^n(s_i-s_{i-1})\right\}^{-\frac{1}{2}}$$
$$\exp\left\{(-1/2)\sum_{i=1}^n(\eta_i-\eta_{i-1})^2/(s_i-s_{i-1})\right\}.$$

Therefore we have the desired result by (3.9).

Finally we evaluate conditional Wiener integral $E^w(Z|X)$ of a real valued Wiener integrable functional Z defined by $Z(X)=\int_{[0,t]}(x(u))^2\,dm_L(u)$ conditioned by X as in Theorem 3.1 and Theorem 3.2.

THEOREM 3.3. *For $x\in C(T)$, let $X(x)=(x(s_1), \ldots, x(s_n))$ and $Z(x)=\int_{[0,t]}(x(u))^2\,dm_L(u)$ where $0=s_0<s_1<\ldots<s_n=t$. Then the conditional Wiener integral of Z given X is*

$$E^w(Z|X)(\eta_1,\ldots,\eta_n)=(1/6)\sum_{i=1}^n(s_i-s_{i-1})$$
$$\{(s_i-s_{i-1})+2(\eta_i^2+\eta_{i-1}\eta_i+\eta_{i-1}^2)\}$$

where $x(s_i)=\eta_i$ for $i=0, 1, \ldots n$.

Proof. It is trivial that $E^w(Z|X)$ exists. By Proposition 3.2, a version of $E^w(Z|X)\dfrac{dP_X}{dm_L}$ is given by

(3.12) $$E^w(Z|X)(\eta_1,\ldots,\eta_n)\frac{dP_X}{dm_L}(\eta_1,\ldots,\eta_n)=\sum_{i=1}^n\lim_{(r_1,\cdots,r_n)\to(0,\cdots,0)}K_i$$

where $K_i=E^w[J\binom{r_1,\ldots,r_n}{\eta_1,\ldots,\eta_n}(x(s_1),\ldots,x(s_n))\int_{[s_i-1,s_i]}(x(u))^2\,dm_L(u)]$ for our Z and X. Thus by (2.5) and (3.7) we have

$$K_i=(1/2^nr_1\ldots r_n)\left\{(2\pi)^n\prod_{i=1}^n(s_i-s_{i-1})\right\}^{-\frac{1}{2}}$$

(3.13) $$\int_{(s_i-1,s_i)}[(u-s_{i-1})(s_i-u)/(s_i-s_{i-1})+\delta^2]$$
$$\left\{\int_{\times_{i=1}^X[\eta_i-r_i,\eta_i+r_i]}\exp\left\{(-1/2)\sum_{i=1}^n(\alpha_i-\alpha_{i-1})^2/(s_i-s_{i-1})\right\}d\alpha_1\ldots d\alpha_n\right\}dm_L(u)$$

where $\delta=((s_i-u)\alpha_{i-1}+(u-s_{i-1})\alpha_i)/(s_i-s_{i-1})$. By (3.12), (3.13), (3.2), and (3.9), the Theorem 3.3 follows.

REMARK. Let $X(x)=x(t)=\xi\in\mathbf{R}^1$ for $x\in C(T)$ and Z be given as in Theorem

3.3. Then $E^w(Z|X)(\xi)=t^2/6+t\xi^2/3$. This shows that Theorem 3.3 is an extension of [4, Example 2, p. 631].

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