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FIXED POINTS ON NONCOMPACT AND NONCONVEX SETS

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1. Introduction

Let X be a Banach space, and let $B(X)$ (resp. $CB(X)$, $K(X)$, $CV(X)$) denote the family of all nonvoid (resp. closed bounded, compact, convex) subsets of X. The Kuratowski measure of noncompactness is defined by the mapping $\alpha_K : B(X) \to R_+$ with $\alpha_K(A) = \inf\{r>0 \mid A$ can be covered by a finite number of sets with diameter less than $r\}$. Then the following properties are valid (see [3, 4]).

(M$_1$) $\alpha_K(A)=0$ implies $\bar{A}\in K(X)$.
(M$_2$) $A\in K(X)$ implies $\alpha_K(A)=0$.
(M$_3$) $\alpha_K(\bar{A})=\alpha_K(A)$ for all $A\in B(X)$.
(M$_4$) $\alpha_K(\overline{\text{co}}A)=\alpha_K(A)$ for all $A\in B(X)$.
(M$_5$) $A\subset B$ implies $\alpha_K(A)\leq\alpha_K(B)$ for all $A, B\in B(X)$.
(M$_6$) $\alpha_K(A\cup B)=\max\{\alpha_K(A), \alpha_K(B)\}$ for all $A, B\in B(X)$.
(M$_7$) If $A_n\in B(X)$, $A_{n+1}\subset A_n$, $n\in N$ and $\alpha_K(A_n)\to 0$ as $n\to\infty$, then $\cap A_n\neq\phi$ and $\alpha_K(\cap A_n)=0$,

where $\bar{A}$ and co A denote the closure and the convex hull of A respectively. It follows immediately that (M$_4$) implies (M$_3$) and (M$_6$) implies (M$_5$).

Also, the Eisenfeld-Lakshmikantham measure of nonconvexity is defined by the mapping $\beta_{EL} : B(X) \to R_+$ with $\beta_{EL}(A)=H(A, \text{co}A)$, where H is the Hausdorff metric. Then the following properties hold (see [2, 6]).

(C$_1$) $\beta_{EL}(A)=0$ implies $\bar{A}\in CV(X)$.
(C$_2$) $\beta_{EL}(\bar{A})=\beta_{EL}(A)$ for all $A\in B(X)$.
(C$_3$) $\beta_{EL} : (CB(X), H)\to R_+$ is continuous.

We say that α (resp. β): $B(X) \to R_+$ is a measure of noncompactness (resp. nonconvexity) if α (resp. β) satisfies some of (M$_1$) – (M$_7$) (resp. (C$_1$) – (C$_3$)).

The purpose of this paper is to give various fixed point theorems using the measure of noncompactness and nonconvexity, which are generalizations of Darbo [1], Eisenfeld-Lakshmikantham [2], Rus [6] and Sadovskii [7].

2. Fixed point theorems

Throughout this paper, let X be a Banach space and Y be a nonvoid closed bounded subset of X. A function $\phi : R_+ \to R_+$ is called a comparison function if ϕ is increasing, $\phi(0)=0$ and $\phi^n(r) \to 0$ as $n\to\infty$ for each $r\in R_+$. Let α (resp. β)

be a measure of noncompactness (resp. nonconvexity).

A continuous mapping $f: Y \to Y$ is said to be α-condensing (resp. β-contractive) if $\alpha(f(A)) < \alpha(A)$ (resp. $\beta(f(A)) < \beta(A)$) whenever $\alpha(A) > 0$ (resp. $\beta(A) > 0$) for all $A \in B(Y)$ such that $f(A) \subset A$. Also, f is said to be (α, ϕ)-contraction if $\alpha(f(A)) \leq \phi(\alpha(A))$ for all $A \in B(Y)$ such that $f(A) \subset A$. In a similar way we define a (β, ϕ)-contraction. First we have the following theorem.

THEOREM 1. *Suppose that Y is compact and β satisfies* (C_1). *If a continuous mapping $f: Y \to Y$ is β-contractive, then f has a fixed point.*

Proof. Since Y is compact, $Y_\infty = \bigcap_{n=1}^{\infty} f^n(Y)$ is nonvoid and compact. It is easy to see that f maps Y_∞ onto Y_∞. Since f is β-contractive, $\beta(Y_\infty) = 0$. Therefore the theorem follows from Schauder's fixed point theorem.

Using Theorem 1, we have the following generalization of [2] and [7].

THEOREM 2. *Suppose that α satisfies* (M_1), (M_2), (M_3) *and* (M_6), *and β satisfies* (C_1). *If a continuous mapping $f: Y \to Y$ is α-condensing and β-contractive, then f has a fixed point.*

Proof. Let $\mathcal{A}$ be the family of all nonvoid closed subsets of Y which is invariant under f and contains a fixed element $y \in Y$. Then by Zorn's lemma, $\mathcal{A}$ has a minimal element A. Now we claim that $\alpha(A) = 0$. Suppose that $\alpha(A) > 0$. Let $A_1 = \overline{f(A)} \cup \{y\}$. Then $A_1 \in \mathcal{A}$ and $A_1 \subset A$, which shows that $A_1 = A$. Since f is α-condensing, $\alpha(A) = \alpha(A_1) = \max\{\alpha(\overline{f(A)}), \alpha(fy)\} = \alpha(f(A)) < \alpha(A)$ by (M_2), (M_3) and (M_6), which is a contradiction. Therefore $\alpha(A) = 0$, and A is compact by (M_1). Therefore the result follows from Theorem 1.

Next, we have the following generalization of [1], [2] and [6].

THEOREM 3. *Suppose α satisfies* (M_1), (M_3) *and* (M_7), *β satisfies* (C_1), *and ϕ is a comparison function. If $f: Y \to Y$ is continuous, (α, ϕ)-contraction and β-contractive, then f has a fixed point.*

Proof. Let $Y_1 = \overline{f(Y)}$ and $Y_{n+1} = \overline{f(Y_n)}$. Therefore $f(Y_n) \subset Y_n$ and $\alpha(Y_n) \leq \phi^n(\alpha(Y)) \to 0$ as $n \to \infty$. Thus by (M_7), $Y_\infty = \bigcap Y_n \neq \phi$ and $\alpha(Y_\infty) = 0$. By (M_1), Y_∞ is compact, and the result follows from Theorem 1.

Finally, if X is reflexive or $\overline{\text{co}}\, Y$ is weakly compact, then we have the following generalization of [2].

THEOREM 4. *Suppose that α satisfies* (M_1), (M_2), (M_4) *and* (M_6), *and $f: Y \to Y$ is continuous, α-condensing and (β_{EL}, ϕ)-contraction, where ϕ is a comparison function. If $\overline{\text{co}}\, Y$ is weakly compact, then f has a fixed point.*

Proof. Let $Y_1 = \overline{f(Y)}$, and $Y_{n+1} = \overline{f(Y_n)}$. Choose $y_n \in Y_n$. Since $\overline{\text{co}}\, Y$ is weakly compact, we may assume that y_n converges weakly to an element $y \in X$. Since $\beta_{EL}(Y_n) \leq \phi^n(\beta_{EL}(Y))$, $d(y, Y_n) \leq \beta_{EL}(Y_n) \to 0$ as $n \to \infty$. This shows that $y \in \bigcap Y_n = Y_\infty$, and $\beta_{EL}(Y_\infty) = 0$. Therefore Y_∞ is nonvoid closed and convex, which is invariant under f, and the result follows from [5].

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