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GROUPOID AS A COVERING SPACE

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1. Introduction

Let X be a topological space. We consider a groupoid G over X and the quotient groupoid G/N for any normal subgroupoid N of G. The concept of groupoid (topological groupoid) is a natural generalization of the group (topological group). An useful example of a groupoid over X is the fundamental groupoid πX whose object group at $x\in X$ is the fundamental group $\pi(X, x)$.

It is known [5] that if X is locally simply connected, then the topology of X determines a topology on πX so that it becomes a topological groupoid over X, and a covering space of the product space $X\times X$.

In this paper the concept of the locally simple connectivity of a topological space X is applied to the groupoid G over X. That concept is defined as a term '1-connected local subgroupoid' of G. Using this concept we topologize the groupoid G so that it becomes a topological groupoid over X. With this topology the connected groupoid G is a covering space of the product space $X\times X$. Furthermore, if $ob(\tilde{G})=\tilde{X}$ is a covering space of X, then the groupoid $\tilde{G}$ is also a covering space of the groupoid G.

Since the fundamental groupoid πX of X satisfying a certain condition has an 1-connected local subgroupoid, πX can always be topologized. In this case the topology on πX is the same as that of [5].

In section 4 the results on the groupoid G are generalized to the quotient groupoid G/N. For any topological groupoid G over X and normal subgroupoid N of G, the abstract quotient groupoid G/N can be given the identification topology, but with this topology G/N need not be a topological groupoid over X [4]. However the induced topology $\mathscr{C}(H)$ on G makes G/N (with the identification topology) a topological groupoid over X.

A final section is related to the covering morphism. Let G_1 and G_2 be groupoids over the sets X_1 and X_2, respectively, and $\phi : G_1 \longrightarrow G_2$ be a covering epimorphism. If X_2 is a topological space and G_2 has an 1-connected local subgroupoid, then we can topologize X_1 so that $ob(\phi) : X_1 \longrightarrow X_2$ is a covering map and $\phi : G_1 \longrightarrow G_2$ is a topological covering morphism.

2. Preliminaries

A groupoid G over a set X is a category in which every morphism is invertible and $ob(G)=X$. For each x, y in X the set of morphisms in G from x to y is denoted by $G(x, y)$. A topological groupoid G over X is a groupoid over the topological space X such that all the structure functions

(1) the initial and fianl maps $\partial_0, \partial_1 : G \longrightarrow X$,

(2) the unit map $u : X \longrightarrow G$, $x \longrightarrow 1_x$,

(3) the composition map $\theta : G\tilde{\times}G \longrightarrow G$, $(a, b) \longrightarrow ba$, whose domain is the set of (a, b) such that $\partial_1(a)=\partial_0(b)$,

(4) the inverse map $G \longrightarrow G$, $a \longrightarrow a^{-1}$

are continuous. Thus the groupoid is a natural generalization of the group, and the topological groupoid is also a natural generalization of the topological group. If x is an object of G, then under the composition the set $G(x, x)$ is a group, written $G\{x\}$, and called the object group, or vertex group, of G at x. As an useful example of a groupoid over the topological space X, we can consider the fundamental groupoid πX whose object group at $x \in X$ is the fundamental group $\pi(X, x)$.

A groupoid G is called connected if $G(x, y)$ is nonempty, and called 1-connected if $G(x, y)$ has exactly one element, for all objects x, y of G. A topological groupoid G is called locally trivial if each x in $ob(G)$ has a neighborhood U such that there is a continuous function $\lambda : U \longrightarrow G$ such that $\lambda(y) \in G(x, y)$ for all $y \in U$.

Let G be a groupoid. Then $H \subset G$ is said to be subgroupoid of G if H is a subcategory of G which is also a groupoid. A subgroupoid N is called wide in G if N has the same objects as G, and called normal if N is wide in G and for all objects x, y of G and $g \in G(x, y)$ we have

$$g^{-1}N\{y\}g=N\{x\}.$$

In such case the quotient groupoid G/N is defined [2].

A morphism $\phi : G_1 \longrightarrow G_2$ of groupoids is simply a functor. For each object x of G the star of x in G, denoted by $St(G, x)$, is the union of the sets $G(x, y)$ for all object y of G. Thus $St(G, x)$ consists of all elements of G with initial point x. A morphism $\phi : \tilde{G} \longrightarrow G$ of groupoids is called a covering morphism if for each oject $\tilde{x}$ of $\tilde{G}$ the restriction of ϕ

$$St(\tilde{G}, \tilde{x}) \longrightarrow St(G, \phi(\tilde{x}))$$

is bijective; specially, ϕ is called a covering epimorphism if ϕ is a surjective covering morphism. Furthermore, we say that ϕ is a topological covering morphism if for each object $\tilde{x}$ of $\tilde{G}$ the restriction of ϕ

$$St(\tilde{G}, \tilde{x}) \longrightarrow St(G, \phi(\tilde{x}))$$

is a homeomorphism. (See [2] and [3]).

3. Groupoids

DEFINITION 3.1. Let G be a groupoid over a topological space X. $H \subset G$ is said to be an 1-connected local subgroupoid of G if each $x \in X$ has a neighborhood

U such that HU is an 1-connected subgroupoid of G, where $HU=\bigcup_{x,y\in U} H(x,y)$.

For $x\in X$, denote $\mathscr{U}(H,x)$ the family of neighborhoods U of x such that HU is an 1-connected subgroupoid of G.

Every groupoid G need not have an 1-connected local subgroupoid, but has an 1-connected (local) subgroupoid if G is connected and X is finite.

THEOREM 3.2. *Let G be a connected groupoid over a topological space X such that X is finite. Then there exists an 1-connected (local) subgroupoid H of G.*

Proof. If the space X consists of one element, then the theorem is clear. We assume that the theorem is satisfied when the space X has $n$elements. Suppose that X has $n+1$ elements, say, $X=\{x_1, \cdots\cdots, x_n, x_{n+1}\}$. Let $A=\{x_1, \ldots, x_n\}$. Then GA is a subgroupoid of G. And there exists an 1-connected (local) subgroupoid L of GA by the assumption. Since G is connected we can choose an element $a^i_{n+1}\in G(x_i, x_{n+1})$ for each $i=1,2,\ldots, n$. With these elements we define an 1-connected (local) subgroupoid H of G as follows;

$$H(x_i, x_j)=L(x_i, x_j) \text{ if } x_i\neq x_{n+1} \text{ and } x_j\neq x_{n+1},$$
$$H(x_i, x_{n+1})=\{a_n{}^n{}_{+1}a_n{}^i \mid a_n{}^i\in L(x_i, x_n)\},$$
$$H(x_{n+1}, x_i)=\{a_i{}^n(a_n{}^n{}_{+1})^{-1} \mid a_i{}^n\in L(x_n, x_i)\},$$
$$H(x_{n+1}, x_{n+1})=\{1_{x_{n+1}}\}.$$

Then it is clear that H is an 1-connected (local) subgroupoid of G.

Using the 1-connected local subgroupoid H of G, we topologize the groupoid G as follows. Given $a\in G$, choose elements U and V of $\mathscr{U}(H,\partial_0(a))$ and $\mathscr{U}(H, \partial_1(a))$, respectively. Let $H(U,a,V)$ be defined by

$$\{cab \mid b\in HU,\ c\in HV\}.$$

Then the set of forms of $H(U,a,V)$ constitutes a basis for a topology on G. This topology will be called the induced topology by H, and denoted by $\mathscr{T}(H)$. Throughout this section we assume that G is a groupoid over the topological space X with the induced topology $\mathscr{T}(H)$ by an 1-connected local subgroupoid H of G.

THEOREM 3.3. *G is a locally trivial topological groupoid over X with (topologically) discrete object groups.*

Proof. First we prove that G is a topological groupoid over X. Only a proof of the continuity of the composition map $\theta : G\tilde{\times}G \longrightarrow G$ is sketched. The proof of continuity of the other maps are similiar. Let $\theta(a,b)=ba$ for $(a,b)\in G\tilde{\times}G$, and W be any neighborhood of ba in G. Then there exist $U\in\mathscr{U}(H, \partial_0(a))$ and $V\in\mathscr{U}(H, \partial_1(b))$ such that $H(U, ba, V)\subset W$. Let $D\in\mathscr{U}(H,\partial_1(a))$. Then $H(U,a,D)$ is a neighborhood of a and $H(D,b,V)$ is a neighborhood of b. Furthermore, $\theta((H(U, a, D)\times H(D, b, V))\cap(G\tilde{\times}G)\subset H(U, ba, V)\subset W$. In fact, if $(\gamma,\delta)\in(H(U, a, D)\times H(D, b, V))\cap(G\tilde{\times}G)$, then $\gamma=caa'$ and

$\delta = b'bd$ for $a' \in HU$, $b' \in HV$, and $c, d \in HD$ such that $\partial_1(c) = \partial_0(d)$. Hence $\theta(\gamma, \delta) = \delta\gamma = b'bdcaa' = b'baa' \in H(U, ba, V)$. Thus the composition map θ is continuous.

Second, G is locally trivial. For $x \in X$, choose an element U of $\mathscr{U}(H, x)$. Define a map $\lambda: U \longrightarrow G$ by $\lambda(y) \in H(x, y)$. Then λ is a well-defined and continuous map since H is an 1-connected local subgroupoid of G.

Finally, $G\{x\}$ has the discrete topology for each $x \in X$. If $a \in G\{x\}$, then the intersection of $G\{x\}$ with a basic neighborhood of a is $\{a\}$. So $G\{x\}$ has the discrete topology.

Let $\mathscr{L}(G)$ be the family of all 1-connceted local subgroupoids of G. Then the relation between the induced topologies on G by the elements of $\mathscr{L}(G)$ is described as follows.

THEOREM 3.4. *Let H_1 and H_2 be in $\mathscr{L}(G)$, and $H_1 \cap H_2$ in $\mathscr{L}(G)$. Then $\mathscr{T}(H_1) = \mathscr{T}(H_2)$.*

Proof. Let $W \in \mathscr{T}(H_1)$ and $a \in W$. Then there exist $U \in \mathscr{U}(H_1, \partial_0(a))$ and $V \in \mathscr{U}(H_1, \partial_1(a))$ such that $H_1(U, a, V) \subset W$. Choose two elements U' and V' of $\mathscr{U}(H_2, \partial_0(a))$ and $\mathscr{U}(H_2, \partial_1(a))$, respectively, such that $U' \subset U$ and $V' \subset V$. Then $H_2(U', a, V') \subset H_1(U, a, V) \subset W$ since $H_1 \cap H_2 \in \mathscr{L}(G)$. Hence we have $W \in \mathscr{T}(H_2)$. The converse is similiar.

COROLLARY 3.5. *Let H_1 and H_2 be in $\mathscr{L}(G)$, and $H_1 \subset H_2$. Then $\mathscr{T}(H_1) = \mathscr{T}(H_2)$.*

Any groupoid G need not have an 1-connected local subgroupoid but every fundamental groupoid πX of X satisfying a certain condition has an 1-connected local subgroupoid.

THEOREM 3.6. *Let πX be the fundamental groupoid of a topological space X. Suppose that there is a covering $\mathscr{U}$ of X whose members are null homotopic open subsets of X such that for any $U, V \in \mathscr{U}$, if $U \cap V \neq \phi$, then $U \cup V$ is contained in some null homotopic open subset of X. Then there exists an 1-connected local subgroupoid of πX.*

Proof. Let H be the set of all homotopy equivalence classes of paths in elements of $\mathscr{U}$. For each $x \in X$ there is a member U of $\mathscr{U}$ containing x. Let y, z be in U. Since there exists a path α in U from y to z, the set $H(y, z)$ is nonempty. Each element of $H(y, z)$ is the homotopy equivalence class of some path β in a member V of $\mathscr{U}$ containing y and z. Since $U \cup V$ is contained in some null homotopic open subset of X, β and α are homotopic. Thus the set $H(y, z)$ has exactly one element, and so HU is 1-connected.

It is not hard to show that HU is a subgroupoid of πX. Consequently, H is an 1-connected local subgroupoid of πX.

THEOREM 3.7. *Let G be a connected groupoid over X. Then G is a covering* *space of the product space $X\times X$. Morever if the cardinality of $G(x,y)$, $x,y\in X$, is n, then G is the n-fold covering space of $X\times X$.*

Proof. Define a map $p: G\longrightarrow X\times X$ by the equation

$$p(a)=(\partial_0(a),\ \partial_1(a)),\ a\in G.$$

Then p is a well-defined and continuous map since the initial and final maps are continuous. It is clear that p is surjective by the connectivity of G.

For any basic neighborhood $H(U, a, V)$ of a in G, $\partial_0(H(U, a, V))=U$ and $\partial_1((H(U,a,V))=V$. Hence $p(H(U,a,V))=U\times V$ is open in $X\times X$. Therefore p is and open map.

Let x, y be in X, and $U\in\mathscr{U}(H,x)$ and $V\in\mathscr{U}(H,y)$. Then $p^{-1}(U,V)=\bigcup_{a\in G(x,y)} H(U,a,V)$. Since HU and HV are 1-connected, $H(U,a,V)\cap H(U, b, V)=\phi$ if $a\neq b$, and for each $a\in G(x,y)$ the restriction of p

$$H(U,a,V)\longrightarrow U\times V$$

is bijective. Consequently, p is a covering map.

By definition, $p^{-1}(x,y)=G(x,y)$ for $x,y\in X$. Since G is connected, $G(x,y)$ and $G(z,w)$ are 1-1 correspondence for any $x,y,z,w\in X$. Hence p is a covering map determined by the cardinality of $G(x,y)$ for $x,y\in X$.

If we consider the subspace $St(G,x)$ of G, then we have the following corollary immediately.

COROLLARY 3.8. *Let G be a connected groupoid over X. For each x in X, the subspace $St(G,x)$ of G is a covering space of X based at x.*

THEOREM 3.9. *Let $\tilde{G}$ and G be groupoids over $\tilde{X}$ and X, respectively. If $\phi: \tilde{G}\longrightarrow G$ is a morphism of groupoids such that $ob(\phi):\tilde{X}\longrightarrow X$ is a covering map, then ϕ is also a covering map.*

Proof. Consider the following diagram

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\ \phi\ } & G \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \tilde{X}\times\tilde{X} & \xrightarrow[\ \phi\times\phi\ ]{} & X\times X \end{array}$$

where p_1 and p_2 are the covering maps considered in Theorem 3.7, and the map $\phi\times\phi$ is defined by $(\phi\times\phi)(x,y)=(\phi(x),\ \phi(y))$ for $x,y\in\tilde{X}$. Then the above diagram commutes. Hence the morphism $\phi: \tilde{G}\longrightarrow G$ is also a covering map.

4. Quotient groupoids

Let G be a groupoid over the topological space X, and N be a normal subgroupoid of G. If H is an 1-connected local subgroupoid of G, then we can easily see that H/N is also an 1-connected local subgroupoid of the quotient groupoid G/N. Hence we can consider the induced topology $\mathscr{T}(H/N)$ on the quotient groupoid G/N. Throughout this section we assume that the groupoid G has the induced topology $\mathfrak{T}(H)$, and the quotient groupoid G/N has the induced topology $\mathfrak{T}(H/N)$.

THEOREM 4.1. *G/N is a locally trivial topological groupoid over X with (topologically) discrete object groups.*

Proof. It is similiar to Theorem 3.3.

THEOREM 4.2. *Let G be a connected groupoid over X, and N be a normal subgroupoid of G. Then G/N is a covering space of the product space $X\times X$. Moreover if the cardinality of $G\{x\}/N\{x\}$, $x\in X$ is n, then G/N is the n-fold covering space of $X\times X$.*

Proof. It is similiar to Theorem 3.7.

COROLLARY 4.3. *Let G be a connected groupoid over X, and N be a normal subgroupoid of G. Then the subspace $St(G/N, x)$ of G/N is a covering space of X based at x.*

THEOREM 4.4. *Let $q: G\longrightarrow G/N$ be the quotient map. Then q is a continuous and open map.*

Proof. Let W be a neighborhood of $q(a)$, $a\in G$. There exist $U\in N(H/N, \partial_0(a))$ and $V\in\mathscr{U}(H/N, \partial_1(a))$ such that $H/N(U, q(a), V)\subset W$. $H(U, a, V)$ is a neighborhood of a and $q(H(U, a, V))\subset H/N(U, q(a), V)\subset W$. Thus q is continuous.

Let W be an open subset of G. For any $a\in W$ there exist $U\in\mathscr{U}(H, \partial_0(a))$ and $V\in\mathscr{U}(H, \partial_1(a))$ such that $H(U, a, V)\subset W$. Then $q(a)\in H/N(U, q(a), V)$ $=q(H(U, a, V))\subset q(W)$ since N is normal. Hence $q(W)$ is an open subset of G/N. Consequently q is an open map.

LEMMA 4.5. *Let $q: G\longrightarrow G/N$ be the quotient map. If $\mathfrak{T}$ is the identification topology on G/N with respect to q, then $\mathfrak{T}=\mathfrak{T}(H/N)$.*

Proof. Since q is continuous, $\mathfrak{T}(H/N)\subset\mathfrak{T}$. Let $W\in\mathfrak{T}$, and $a\in W\subset G/N$. Then we have $q^{-1}(a)\subset q^{-1}(W)\in\mathfrak{T}(H)$. Choose an element b in $q^{-1}(a)$. Then there exist $U\in\mathscr{U}(H, \partial_0(b))$ and $V\in\mathscr{U}(H, \partial_1(b))$ such that $H(U, b, V)\subset q^{-1}(W)$. Consequently we get $H/N(U, a, V)=q(H(U, b, V))\subset qq^{-1}(W)\subset W$. Hence we proved that $W\in\mathfrak{T}(H/N)$.

For any topological groupoid G and normal subgroupoid N of G, the quotient groupoid G/N can be given the identification topology, but the proof of the continuity used in the group case breaks down [4]. However the induced topology $\mathfrak{T}(H)$ on G makes G/N (with the identification topology) a topological groupoid over X.

THEOREM 4.6. *Let G/N be topologized by the identification map $q: G\longrightarrow G/N$. Then G/N is a locally trivial topological groupoid over X with (topological-*

ly) discrete object groups.

Proof. By Lemma 4.5, the identification topology on G/N is the same as the induced topology $\mathscr{T}(H/N)$. Hence G/N is a topolgical groupoid over X by Theorem 4.1.

5. Covering morphisms

Let G_1 and G_2 be groupoids over the sets X_1 and X_2, respectively, and $\phi : G_1 \longrightarrow G_2$ be a covering epimorphism. If X_2 is a topological space and G_2 has an 1-conneced local subgoupoid H_2 of G_2, thetn we topologize X_1 so that $\phi : X_1 \longrightarrow X_2$ is a covering map and $\phi : G_1 \longrightarrow G_2$ is a topological covering morphism.

For $x \in X_1$, choose an element U of $\mathscr{U}(H_2, \phi(x))$. Let $U(x)$ be defined by

$$\{y \in X_1 \mid \text{there exists } a \in G_1(x, y) \text{ such that } \phi(a) \in H_2 U\}.$$

Then we can easily prove that the set of the forms of $U(x)$ constitutes a basis for a topology on X_1.

With this topology on X_1, we have the followings.

THEOREM 5.1. *$\phi : X_1 \longrightarrow X_2$ is a covering map.*

Proof. Let W be any open subset of X_2, and $x \in \phi^{-1}(W)$. Choose an element U of $\mathscr{U}(H_2, \phi(x))$ such that $U \subset W$. Then $U(x)$ is a basic neighborhood of x. Let $y \in U(x)$. Then there exists $a \in G_1(x, y)$ such that $\phi(a) \in H_2 U$. Hence we have $\phi(y) \in U$, and so $y \in \phi^{-1}(U)$. Consequently, we get $U(x) \subset \phi^{-1}(U) \subset \phi^{-1}(W)$ Thus ϕ is a continuous map.

Let V be any open subset of X_1, and $y \in \phi(V)$. Then there exists $x \in V$ such that $\phi(x) = y$. Choose an element U of $\mathscr{U}(H_2, y)$ such that $U(x) \subset V$. Then we can easily see that $U \subset \phi(U(x)) \subset \phi(V)$. Hence ϕ is an open map.

Let $y \in X_2$. Since $\phi : G_1 \longrightarrow G_2$ is a covering epimorphism, there exists $a \in G_1$ such that $\phi(a) = 1_y$. Hence we have $\partial_0(a) = x \in X_1$, and $\phi(x) = y$. So ϕ is surjective.

Let $y \in X_2$, and $U \in \mathscr{U}(H_2, y)$. Then we have $\phi^{-1}(U) = \bigcup_{x \in \phi^{-1}(y)} U(x)$, where the union is disjoint. In fact, if $U(x) \cap U(z) \neq \phi$ for $x, z \in \phi^{-1}(y)$, then there exists $w \in U(x) \cap U(z)$, and so exist $a \in G_1(x, w)$ and $b \in G_1(z, w)$ such that $\phi(a)$, $\phi(b) \in H_2 U$. Since $\phi(x) = \phi(z)$, $\phi(a) = \phi(b)$, and so $a^{-1} = b^{-1}$. Consequently, we have $x = z$.

Finally it is not hard to show that $\phi : U(x) \longrightarrow U$ is bijective for each $x \in \phi^{-1}(y)$.

Now we construct an 1-connected local subgroupoid H_1 of G_1, and using this subgroupoid H_1 we prove that the covering morphism $\phi : G_1 \longrightarrow G_2$ is a topological covering morphism.

LEMMA 5.2. *Let $H_1 = \{a \in G_1 \mid \text{there is } U \in \mathscr{U}(H_2, \partial_0(\phi(a))) \text{ such that } \phi(a) \in H_2 U\}$. Then H_1 is an 1-connected local subgroupoid of G_1.*

Proof. For $x \in X_1$, choose an element U of $\mathcal{U}(H_2, \phi(x))$. Then $U(x)$ is a basic neighborhood of x. It is enough to show that $H_1U(x)$ is an 1-connected subgroupoid of G.

Let $y, z \in U(x)$. Then there exist $a \in G_1(x, y)$ and $b \in G_1(x, z)$ such that $\phi(a)$ and $\phi(b)$ are in H_2U. Since $\phi(y) \in U$, $U \in \mathcal{U}(H_2, \phi(y))$. Now $ba^{-1} \in G_1(y, z)$ and $\phi(ba^{-1}) = \phi(b)\phi(a)^{-1} \in H_2U$. By definition of H_1, $ba^{-1} \in H_1(y, z)$. Hence we have $H_1(y, z) \neq \phi$.

Let $c, d \in H_1(y, z)$. Then $\phi(c)$ and $\phi(d)$ are in H_2U, and so $\phi(c) = \phi(d)$. Since ϕ is a covering morphism, $c = d$. Consequently, $H_1U(x)$ is 1-connected.

Let a and b be two elements of $H_1U(x)$ such that $\partial_0(b) = \partial_1(a)$. Then $a \in H_1(y, z)$ and $b \in H_1(z, w)$ for some $y, z, w \in U(x)$. By definition, there exist $U_1 \in \mathcal{U}(H_2, \phi(y))$ and $U_2 \in \mathcal{U}(H_2, \phi(z))$ such that $\phi(a) \in H_2U_1$ and $\phi(b) \in H_2U_2$. On the other hand $\phi(a)$ and $\phi(b)$ are in H_2U, and $U \in \mathcal{U}(H_2, \phi(y))$. Thus $\phi(ba) = \phi(b)\phi(a) \in H_2U$, and so $ba \in H_1(y, w) \subset H_1U(x)$.

Similarly we can prove that the inverse of each element of $H_1U(x)$ is also in $H_1U(x)$. Hence $H_1U(x)$ is a subgroupoid of G_1.

THEOREM 5.3. *$\phi : G_1 \longrightarrow G_2$ is a topological covering morphism.*

Proof. Let $a \in G_1$, and W be any neighborhood of $\phi(a)$, Then there exist $U_2 \in \mathcal{U}(H_2, \partial_0(\phi(a))$ and $V_2 \in \mathcal{U}(H_2, \partial_1(\phi(a))$ such that $H_2(U_2, \phi(a), V_2) \subset W$. Since $\phi : X_1 \longrightarrow X_2$ is continuous, there exist $U_1 \in \mathcal{U}(H_1, \partial_0(a))$ and $V_1 \in \mathcal{U}(H_1, \partial_1(a))$ such that $\phi(U_1) \subset U_2$ and $\phi(V_1) \subset V_2$. By definition of H_1, $\phi(H_1) \subset H_2$. Hence we get $\phi(H_1(U_1, a, V_1)) \subset H_2(U_2, \phi(a), V_2) \subset W$. Thus ϕ is continuous.

Let W be any open subset of G_1, and $a \in \phi(W)$. Then there exists $b \in W$ such that $\phi(b) = \mathrm{a}$, and exist $U_1 \in \mathcal{U}(H_1, \partial_0(b))$ and $V_1 \in \mathcal{U}(H_1, \partial_1(b))$ such that $H_1(U_1, b, V_1) \subset W$. Furthermore there exist $U_2 \in \mathcal{U}(H_2, \partial_0(a))$ and $V_2 \in \mathcal{U}(H_2, \partial_1(a))$ such that $U_2(\partial_0(b)) \subset U_1$ and $V_2(\partial_1(b)) \subset V_1$. We get $H_2(U_2, a, V_2) \subset \phi(H_1(U_1, b, V_1)) \subset \phi(W)$. Thus $\phi(W)$ is an open subset of G_2. Hence ϕ is an open map. Consequently, we proved that the restriction of ϕ

$$St(G_1, x) \longrightarrow St(G_2, \phi(x))$$

is a homeomorphism for each $x \in X_1$.

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