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NEARLY KAEHLERIAN PRODUCT MANIFOLDS OF TWO ALMOST CONTACT METRIC MANIFOLDS

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Dedicated to Professor Sang-Seup Eum on his 60th birthday

0. Introduction

It is well-known that the most interesting non-integrable almost Hermitian manifold are the nearly Kaehlerian manifolds ([2] and [3]), and that there exists a complex but not a Kaehlerian structure on Riemannian product manifolds of two normal contact manifolds [4].

The purpose of the present paper is to study nearly Kaehlerian product manifolds of two almost contact metric manifolds and investigate the geometrical structures of these manifolds. Unless otherwise stated, we shall always assume that manifolds and quantities are differentiable of class C^∞.

In Paragraph 1, we give brief discussions of almost contact metric manifolds and their Riemannian product manifolds. In paragraph 2, we investigate the perfect conditions for Riemannian product manifolds of two almost contact metric manifolds to be nearly Kaehlerian and the non-existence of a nearly Kaehlerian product manifold of contact metric manifolds. Paragraph 3 will be devoted to a proof of the following;

A conformally flat compact nearly Kaehlerian product manifold of two almost contact metric manifolds is isometric to a Riemannian product manifold of a complex projective space and a flat Kaehlerian manifold.

1. Preliminaries

A $(2n+1)$-dimensional differentiable manifold N is said to be an *almost contact metric manifold* if it admits a tensor field ϕ of type $(1,1)$, a Riemannian metric g, a vector field ξ and a 1-form η such that

$$(1.1)\qquad \begin{cases} \phi^2X=-X+\eta(X)\xi, & \eta(\xi)=1, \\ \phi\xi=0, & \eta(\phi X)=0, \\ g(\phi x,\ \phi Y)=g(X,\ Y)-\eta(X)\eta(Y) & \end{cases}$$

for any vector fields X and Y tangent to N. The distribution D_y, $y\in N$, given by

$$D_y=\{X\in T_y(N)\,|\,\eta(X)=0\}$$

is called a *contact distribution* of N, where $T_y(N)$ indicates the tangent space of N at a point y of N. If an almost contact metric manifold N admits a global 1-form η such that $\eta\wedge(d\eta)^n\neq 0$, then we say that N is a *contact metric manifold*. It is well-known that the contact metric manifold N satisfies the equations

$$d\eta(X,\ Y)=g(x,\ \phi Y)$$

and

$$\nabla_X\xi=-\phi X-\frac{1}{2}(\mathcal{L}_\xi\phi)X, \tag{1.2}$$

where ∇ is the covariant differentiation with respect to g and $\mathcal{L}_\xi$ the operator of Lie derivative with respect to ξ.

Let $M=N_1{}^{2m+1}\times N_2{}^{2n+1}$ be a Riemannian product manifold of two almost contact metric manifolds N_1 and N_2 with structure $(\phi_1,\ g_1,\ \xi_1,\ \eta_1)$ and $(\phi_2,\ g_2,\ \xi_2,\ \eta_2)$ respectively. The manifolds N_1 and N_2 are called *parts* of M.

The projection operators of the tangent space $T_x(M)$, $x\in M$, of M to the tangent spaces $T_x(N_1)$ of N_1 and $T_x(N_2)$ of N_2 will be denoted by P and Q respectively. Then it is easily seen that

$$P+Q=I,\ \ P^2=P,\ \ Q^2=Q,\ \ PQ=QP=0.$$

If we put $F=P-Q$ and we choose a Riemannain metric g of M as

$$g(X,\ Y)=g_1(PX,\ PY)+g_2(QX,\ QY)$$

for any vector fields X and Y tangent to M, then we can easily verify that (g,F) is a Riemanian product structure on M.

Now we define a tensor field J of type $(1,1)$ on M by

$$JX=\phi_1PX+\phi_2QX+\eta_2(QX)\xi_1-\eta_1(PX)\xi_2. \tag{1.3}$$

Then it follows from (1.1) that (g,J) is an almost Hermitian structure on M. It is known [4] that the almost complex structure J is integrable if and only if the structure $(\phi_1,\ \xi_1,\ \eta_1)$ of N_1 and $(\phi_2,\ \xi_2,\ \eta_2)$ of N_2 are both integrable. A vector field X tangent to M is said to be *parallel* if $\nabla_YX=0$ identically for any vector field Y tangent to M.

A submanifold N of M is also said to be *parallel* in M if $\nabla_YX\in T_y(N)$ for any vector fields X tangent to N and Y tangent to M at any point y of N. Since each part $N_\alpha(\alpha=1,2)$ of M is totally geodesic and parallel in M, it follows from the definition of J that

$$\nabla_{PX}\phi_2=\nabla_{QX}\phi_1=0,\ \ \nabla_{PX}\xi_2=\nabla_{QX}\xi_1=0. \tag{1.4}$$

As for the parallelism of the structure vector field ξ on an almost contact metric manifold, we prove the following lemma which will be used later.

LEMMA 1.1. *An almost contact metric manifold N^{2n+1} is a Riemannian product manifold $\bar{N}^{2n}\times R$ if and only if the structure vector field ξ is parallel, where $\bar{N}$ is the maximal integral manifold of the contact distribution and R is that of trajectories of ξ.*

Proof. If N is the Riemannian product manifold $\bar{N}\times R$, then R is parallel in N. Since $g(\nabla_X\xi, \xi)=0$ identically for any vector field X tangent to N, we see that ξ is parallel. Conversely, if ξ is parallel, then we have $g(\nabla_X Y, \xi)=0$ identically for any vector fields X tangent to N and Y belonging to the contact distribution, which shows that the distribution is parallel in N.

2. Nearly Kaehlerian product manifolds

An almost Hermitian manifold M with the almost complex structure J is said to be *nearly Kaehlerian* (or *Tachibana*) if it satisfies

$$(\nabla_X J)Y+(\nabla_Y J)X=0 \tag{2.1}$$

for any vector fields X and Y tangent to M.

Let M be a nearly Kaehlerian product manifold $N_1\times N_2$ of two almost contact metric manifolds N_1 and N_2. Putting $X=PX$ and $Y=QY$ in (2.1) and using (1.3) and (1.4), we obtain

$$\eta_2(QY)\nabla_{PX}\xi_1=\eta_1(PX)\nabla_{QY}\xi_2,$$

which shows that

$$\nabla_{PX}\xi_1=\nabla_{QY}\xi_2=0. \tag{2.2}$$

Hence each part $N_\alpha(\alpha=1,2)$ of M is a Riemannian product manifold $\bar{N}\times R$ by Lemma 1.1. Putting $X=PX$ and $Y=PY$ in (2.1), we have

$$(\nabla_{PX}\phi_1)PY+(\nabla_{PY}\phi_1)PX=0, \tag{2.3}$$

which shows that the part $\bar{N}_1$ of N_1 is nearly Kaehlerian because the restriction of J to $\bar{N}_1$ is ϕ_1. Similarly, putting $X=QX$ and $Y=QY$ in (2.1), we have

$$(\nabla_{QX}\phi_2)QY+(\nabla_{QY}\phi_2)QX=0 \tag{2.4}$$

and the part $\bar{N}_2$ of N_2 is nearly Kaehlerian too.

Conversely, if the equations (2.2), (2.3) and (2.4) are satisfied on the product manifold $M=N_1\times N_2$, then it is easily seen that the equation (2.1) is also satisfied. Thus we have the following

THEOREM 2.1. *A Riemannian product manifold of two almost contact metric manifolds N_1 and N_2 is nearly Kaehlerian if and only if each manifold N_α ($\alpha=1,2$) is a Riemannian product manifold $\bar{N}_\alpha\times R$, where $\bar{N}_\alpha$ and R are the maximal integral nearly Kaehlerian manifold of the contact distribution and an open interval respectively.*

Let K, K_1 and K_2 be the curvature tensors of M, N_1 and N_2 respectively. Then, on the Riemannian product manifold $M=N_1\times N_2$, we have the relations

$$K(PX.\ PX)PZ=K_1(PX,\ PY)PZ,\quad K(QX,QY)QZ=K_2(QX,\ QY)QZ$$

and

$$K(X,\ Y)Z=0 \text{ otherwise}$$

for any vector fields X, Y and Z tangent to M. Since $\nabla_X\xi_1=\nabla_X\xi_2=0$, we obtain

$K(X, Y)\xi_1=K(X, Y)\xi_2=0$. Moreover it follows from (2.3) and (2.4) that $\nabla_{\xi_1}\phi_1=\nabla_{\xi_2}\phi_2=0$. Summing up this result and Theorem 2.1, we can state

THEOREM 2.2. *Let M be a nearly Kaehlerian product manifold of two almost contact metric manifolds. Then M is a Riemannian product manifold $\bar{M}\times R^2$ of a nearly Kaehlerian manifold $\bar{M}$ and a 2-dimensional flat Kaehlerian manifold R^2.*

As for a contact metric part of the Riemannian product manifold, we can state the following.

THEOREM 2.3. *There is no nearly Kaehlerian product manifold of a contact metric manifold and an almost contact metric manifold.*

Proof. Suppose that M is a nearly Kaehlerian product manifold $N_1\times N_2$ of a contact metric manifold N_1 and an almost contact manifold N_2. Then it follows from (1.2) and (2.2) that

$$\nabla_X\xi_1=-\phi_1X-\frac{1}{2}\phi_1(\mathcal{L}_{\xi_1}\phi_1)X=0$$

for any vector field X tangent to M. Since

$$(\mathcal{L}_{\xi_1}\phi_1)X=[\xi_1,\ \phi_1X]-\phi_1[\xi_1,\ X]=0,$$

the above equation is reduced to $\phi_1X=0$ and it is a contradiction.

3. Conformally flat nearly Kaehlerian product manifolds

In this paragraph, we shall investigate some properties of conformally flat nearly Kaehlerian product manifolds of two almost contact metric manifolds.

First of all, we recall here some definitions and a well-known result which will be used later, for an almost Hermitian manifold $(\bar{M}, g)$.

For convenience we write $\|X\|^2=g(X,X)$. If, for any vector fields X, Y and Z tangent to $\bar{M}$ with $g(X, Y)=g(JX, Y)=g(X, Z)=g(JX, Z)=0$ and $\|Y\|=\|Z\|$, it satisfies the relation $\|(\nabla_XJ)Y\|=\|(\nabla_XJ)Z\|$ at point of $\bar{M}$, then $\bar{M}$ is said to be of *constant type* at the point. If it satisfies for all point of $\bar{M}$, then we say that $\bar{M}$ has *pointwise constant type*. If $\bar{M}$ has pointwise constant type and the function $\|(\nabla_XJ)Y\|$ is constant for any vector fields X and Y tangent to $\bar{M}$ with $g(X, Y)=g(JX, Y)=0$ and $\|X\|=\|Y\|=1$, then we say that $\bar{M}$ has *global constant type*. The *holomorphic bisectional curvature* $B_{\bar{M}}$ of $\bar{M}$ is defined by

$$B_{\bar{M}}(X, Y)=\|X\|^{-2}\|Y\|^{-2}g(K_{\bar{M}}(X,\ JX)JY,\ Y)$$

for any vector fields X and Y tangent to $\bar{M}$, where $K_{\bar{M}}$ is the curvature tensor of $\bar{M}$. The following theorem is due to A. Gray [2].

THEOREM A. *Let $\bar{M}$ be a compact Einstein nearly Kaehlerian manifold of pointwise constant type. If $\bar{M}$ has positive sectional curvature and nonnegative holomorphic bisectional curvature, then $\bar{M}$ is isometric to a complex projective* *space or to a 6-dimensional sphere* S^6.

Returning to our nearly Kaehlerian product manifold $M=N_1^{2m+1}\times N_2^{2n+1}$ of two almost contact metric manifolds N_1 and N_2, the *conformal curvature tensor* C of M is given by

$$(3.1)\quad C(X,Y)Z=K(X,Y)Z+\frac{1}{2(m+n)}\{R(X,Z)Y-R(Y,Z)X+g(X,Z)LY$$
$$-g(Y,Z)LX\}-\frac{k}{2(m+n)(2m+2n+1)}\{g(X,Z)Y-g(Y,Z)Y\}$$

for any vector fields X, Y and Z tangent to M, where R is the Ricci curvature, L the Ricci operator and k the scalar curvature of M. If M is conformally flat, that is, the conformal curvature tensor C of M vanishes identically, then, by putting $Y=Z=\xi_\alpha(\alpha=1,2)$ in (3.1), we obtain

$$(3.2)\qquad LX=\frac{k}{2m+2n+1}(X-\eta_\alpha(X)\xi_\alpha)$$

because of $K(X,Y)\xi_\alpha=0$. Since M is a Riemannian product manifold $\bar{M}\times R^2$ of a nearly Kaehlerian manifold $\bar{M}$ and a flat Kaehlerian manifold R^2 by Theorem 2.2, the Ricci operator $L_{\bar{M}}$ of the part $\bar{M}$ of M is given by

$$(3.3)\qquad L_{\bar{M}}X=\frac{k}{2m+2n+1}X$$

for any vector field X tangent to $\bar{M}$. Moreover, we see from (3.1) and (3.3) that the curvature tensor $K_{\bar{M}}$ of $\bar{M}$ is given by

$$(3.4)\quad K_{\bar{M}}(X,Y)Z=\frac{k}{2(m+n)(2m+2n+1)}(g(Y,Z)X-g(X,Z)Y)$$

for any vector fields X, Y and Z tangent to $\bar{M}$. We can verify that the scalar curvature k of M is a non-zero constant. In fact, if $k=0$ identically, then the well-known identity

$$g(K_{\bar{M}}(X,\ Y)Y,\ X)-g(K_{\bar{M}}(X,\ Y)JY,\ JX)=\|(\nabla_XJ)Y\|^2$$

and the equation (3.4) lead to

$$(\nabla_XJ)Y=0,$$

which shows that the part $\bar{M}$ of M is Kaehlerian and hence M itself is Kaehlerian. It is a contradiction. Thus we can state

THEOREM 3.1. *Let M be a conformally flat nearly Kaehlerian product manifold of two almost contact metric manifolds. Then M is a Riemannian product manifold $\bar{M}\times R^2$ of a nearly Kaehlerian manifold $\bar{M}$ of non-zero constant curvature and a 2-dimensional flat Kaehlerian manifold R^2.*

Since the part $\bar{M}$ of M is a space of non-zero constant curvature, it is easily seen that $\bar{M}$ has global constant type and has constant sectional curvature $k/2(m+n)(2m+2n+1)$. For any non-zero vector fields X and Y tangent to M, the holomorphic bisectional curvature $B_{\bar{M}}$ of $\bar{M}$ is given by

$$B_{\bar{M}}(X, Y)=\frac{k}{2(n+m)(2m+2n+1)}\|X\|^{-2}\|Y\|^{-2}(g(X, Y)^2+g(JX, Y)^2).$$

Therefore, if the manifold M has non-negative scalar curvature, then the part $\bar{M}$ of M has positive sectional curvature and positive holomorphic bisectional curvature. Finally we easily find that the dimension of $\bar{M}$ is equal to $4p$, $p=1, 2, \ldots, (m+n)/2$. Applying Theorem A to Theorem 3.1, we can state

THEOREM 3.2. *Let M be a conformally flat compact nearly Kaehlerian product manifold of two almost contact metric manifolds with non-negative scalar curvature. Then M is isometric to a Riemannian product manifold $\bar{M}\times R^2$ of a complex projective space $\bar{M}$ and a 2-dimensional flat Kaehlerian manifold R^2.*

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