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On Quasi-Euler's Differential Equation

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要 約

오일러의 미분 방정식은 보조 방정식을 이용하면 쉽게 일반해를 구할 수 있다. 정의된 유사-오일러 미분 방정식을 풀기 위해 규정된 조건을 만족토록 하면 보조 방정식을 이용하여 일반해를 쉽게 구할 수 있음을 보였다.

1. INTRODUCTION

It is known that the solution of the second order differential uations with constant coefficients is easily obtained by various :thods. But, it is not easy to find the solution of the second ·der differential equations with variable coefficients.

In the case of the second order Cauchy's differential equations ›r Euler's equations), Legendre equations and Bessel's equations :re found as the methods to solve the differential equations. We ıall now consider linear differential equation of the second order ıose coefficients are variables.

The purpose of this note is to present that the general solution : specified differential equation with variable coefficients is :termined by solving the auxiliary equation.

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2. THE EULER'S HOMOGENEOUS DIFFERENTIAL EQUATION

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + a_2 x^{n-2} y^{(n-2)} + \ldots + a_n y = 0$$

is called the Euler's homogeneous differential equation. The general solution of Euler's differential equation can be found by substitutin; x^r for y and its derivatives.

This substitution yields an equation of the form $A(r)x^r = 0$, whe] $A(r)$ is a polynomial of degree n, called the auxiliary equation.

Any r for which $A(r) = 0$ gives a solution x^r.

Example

If n=2, we obtain

$$x^2 y'' + a_1 xy' = a_2 y = 0 \quad (a_1,\ a_2;\ \text{real constants}).$$

Substituting $y = x^r$ and its derivatives, we have

$$x^2 r(r-1)x^{r-2} + a_1 xrx^{r-1} + a_2 x^r = 0.$$

By omitting the common power x, which is not zero, when $x \neq 0$, we ca] get the auxiliary equation

$$r^2 + (a_1 - 1)r + a_2 = 0.$$

If the roots r_1 and r_2 of given equation are different, then two func tions, $y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$ constitute a basis of solutions oi the equation for all x for which these functions are defined.

The corresponding general solution is

$$y = c_1 x^{r_1} + c_2 x^{r_2} \quad (c_1,\ c_2;\ \text{arbitrary constants}).$$

3. QUASI-EULER'S HOMOGENEOUS DIFFERENTIAL EQUATIONS AND EXAMPLES

Euler's differential equation;

$$x^2y'' + a_1xy' + a_2y = 0$$

be expressed in the form of

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{where } P(x) = a_1\frac{1}{x} \text{ and } Q(x) = \frac{a_2}{x^2}.$$

oposition 1] The general solution of Euler's differential equation

$$y'' + P(x)y' + Q(x)y = 0 \quad (P(x) = \frac{a_1}{x}, \quad Q(x) = \frac{a_2}{x^2}) \text{ is}$$

$$y_g = c_1e^{r_1z} + c_2e^{r_2z}, \text{ where } Q(x) = a_2(e^{-\int P(x)dx})^2$$

$$z = \int e^{-\int P(x)dx}\, dx.$$

oof) Note that $y' = \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$,

$$y'' = \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\cdot(\frac{dz}{dx})^2 + \frac{dy}{dz}\cdot\frac{d^2z}{dx^2}$$

en, the equation $y'' + P(x)y' + Q(x)y = 0$ is

$$(\frac{dz}{dx})^2\frac{d^2y}{dz^2} + \frac{d^2z}{dx^2}\cdot\frac{dy}{dz} + P(x)\frac{dz}{dx}\cdot\frac{dy}{dz} + Q(x)y = 0.$$

e resulting equation is

$$(\frac{dz}{dx})^2\frac{d^2y}{dz^2} + (\frac{d^2z}{dx^2} + P(x)\cdot\frac{dz}{dx})\frac{dy}{dz} + Q(x)y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

z is chosen so that $\frac{d^2z}{dx^2} + P(x)\cdot\frac{dz}{dx} = 0$, then, by using of variables separation method

$$\frac{z''}{z'} = - P(x),$$

the primitive may be obtained by integration of

$$\int\frac{z''}{z'}\,dz = -\int P(x)dx,$$

i.e., $z' = e^{-\int P(x)dx}$.

Hence $z = \int e^{-\int P(x)\,dx}dx$.

From the equation (1),

$$\left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2}+Q(x)y = 0 \quad \text{or} \quad \frac{d^2y}{dz^2} + \frac{Q(x)}{\left(\frac{dz}{dx}\right)^2}y = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

Substituting $a_2(e^{-\int P(x)dx})^2$ for $Q(x)$ the equation (2) yields

$$\frac{d^2y}{dz^2} + a_2y = 0.$$

Since this equation is the second order differential equation with constant coefficients, we can easily obtain the general solutio using the auxiliary equation.

If the roots of auxiliary equation $r^2 + a_2 = 0$ are r_1 and r_2, the general solution of Euler's differential equation is

$$y_g = c_1e^{r_1z} + c_2e^{r_2z} \quad (z = \int e^{-\int P(x)dx}dx).$$

[Corollary 1] If a is a negative number, then the auxiliary equatio of Euler's equation has two distinct real roots r_1 and r_2.

Example 1

Solve the equation $x^2y'' + xy' - y = 0$.

From $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$,

$$Q(x) = -\frac{1}{x^2} = -(e^{-\int\frac{1}{x}dx})^2.$$

ce the auxiliary equation $r^2 - 1 = 0$ has two distinct real roots $= 1$ and $r_2 = -1$,

$$z = \int e^{-\frac{1}{x}\,dx}dx$$

$$= \ell nx.$$

ce

$$y = c_1e^z + c_2e^{-z}$$

$$= c_1e^{\ell nx} + c_2e^{-\ell nx}$$

$$= c_1x + c_2x^{-1}.$$

s general solution is the same as the solution obtained by Euler's hod.

orollary 2] If a_2 is a positive number, then the auxiliary equation Euler's differential equation has two distinct imaginary roots r_1 d r_2.

ample 2

Solve the equation $x^2y'' + xy' + y = 0$.

From $y'' + \frac{1}{x}y' + \frac{1}{x^2}y = 0$,

$$Q(x) = (e^{-\int\frac{1}{x}dx})^2$$

The auxiliary equation $r^2 + 1 = 0$ has two distinct imaginary roots, $r_1 = i$ and $r_2 = -i$ and

$$z = \int e^{-\int \frac{1}{x} dx} dx$$

$$= \ln x.$$

Hence

$$y_g = c_1 e^{iz} + c_2 e^{-iz}$$

$$= c_1 e^{i\ln x} + c_2 e^{-i\ln x}$$

$$y_g = c_1 \cos(\ln x) + c_2 \sin(\ln x).$$

This general solution is the same as one obtained by Euler's method.

[Theorem 2] The general solution of Quasi-Euler's differential equation

$$x^n y'' + \frac{n}{2} x^{n-1} y' + ay = 0 \quad \text{or} \quad y'' + P(x)y' + Q(x)y = 0,$$

where $P(x) = \frac{n}{2} \cdot \frac{x^{n-1}}{x^n}$ and $Q(x) = \frac{a}{x^n}$, is

$$y_g = c_1 e^{r_1 z} + c_2 e^{r_2 z},$$

provided that $Q(x) = a(e^{-\int P(x)dx})^2$ and .

$$z = \int e^{-\int P(x)dx} dx.$$

(Proof) From [proposition 1], note that the equation

$y'' + P(x)y' + Q(x)y = 0$ is

$$\left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dz^2} + \left(\frac{d^2z}{dx^2} + P(x)\cdot\frac{dz}{dx}\right) \frac{dy}{dz} + Q(x)y = 0.$$

z is chosen so that $\frac{d^2z}{dx^2} + P(x)\cdot\frac{dy}{dx} = 0$, then

$$z = \int e^{-\int P(x)dx}dx, \quad Q(x) = \frac{a}{x_n} = a(\exp(-\int\frac{n}{2}\cdot\frac{x^{n-1}}{x^n}dx))^2.$$

r_1 and r_2 are the roots of the auxiliary equation $r^2 + a = 0$, the
ieral solution of Quasi-Euler's differential equation

$$x^n y'' + \frac{n}{2}x^{n-1}y' + ay = 0 \text{ is}$$

$$y_g = c_1e^{r_1z} + c_2e^{r_2z}, \text{ where } z = \int e^{-\int P(x)dx}dx.$$

orollary 3] A differential equation

$$(x^n + b)y'' + \frac{n}{2}x^{n-1}y' + ay = 0$$

s the general solution

$$y_g = c_1e^{r_1z} + c_2e^{r_2z}, \quad \text{where}$$

$$z = \int e^{-\int P(x)dx}dx \text{ and } Q(x) = a(e^{-\int P(x)dx})^2.$$

Example 3

Solve the Quasi-Euler's differential equation

$$x^6y'' + 3x^5y' - y = 0.$$

'he auxiliary equation $r^2 - 1 = 0$ has two distinct real roots,

$r_1 = 1$ and $r_2 = -1$. Therefore

$$z = \int e^{-3\int\frac{1}{x}dx}dx$$

$$Q(x) = -\frac{1}{x^6} = -(e^{-3\int\frac{1}{x}dx})^2.$$

Hence

$$y_g = c_1e^{z} + c_2e^{-z}$$
$$= c_1e^{-\frac{1}{2}x^{-2}} + c_2e^{\frac{1}{2}x^{-2}}$$

[Corollary 4] (General Case)

The differential equation

$$f(x)y'' - g(x)y' + ay = 0 \quad (\text{or } y'' + P(x)y' + Q(x)y = 0,$$

$$\text{where} \quad P(x) = -\frac{g(x)}{f(x)} \quad \text{and} \quad Q(x) = \frac{a}{f(x)})$$

has the general solution $y_g = c_1e^{r_1 z} + c_2e^{r_2 z}$

if $Q(x) = a(e^{-\int P(x)dx})^2$ and $z = \int e^{-\int P(x)dx}dx$.

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