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THE EXISTENCE AND UNIQUENESS OF THE SOLUTION OF A BOUNDARY VALUE PROBLEM FOR A VECTOR DIFFERENTIAL EQUATION OF nth ORDER IN BANACH SPACE

By M. Sobeih and G. Hamad

1. Introduction:

The problem which we study here, is firstly suggested by Niclibork [4]. The problem deals with a projectile with a given initial velocity hits a target. It was formulated as an initial boundary value problem with a system of two differential equations of second order. In 1966 Perov and Makhmoudov [6] generalized the problem for a system of n-equations using vector notations. Many other different generalizations for this problem are suggested and studied in [1, 2, 7, 8, 9].

In this work, we mainly adjoin the works of Bagirian [1] and Sobeih [7], getting a more generalized formulation to the same problem.

2. Formulation of the problem:

Find the solution $(x(t),\ t_*)$ for the following problem:

$$x^{(n)}=f(t,\ x,\ x^{(1)},\ \cdots,\ x^{(n-1)})\ ;\ t\in[0,\ T], \tag{1}$$

$$x^{(i)}(0)=0\ ;\ i\neq i_0\ ;\ 0<i_0\leq n-1\ \ (i=0,\ n-1\ ;\ n\geq 2), \tag{2}$$

$$\|x^{(i_0)}(0)\|=v>0, \tag{3}$$

and

$$\sum_{j=0}^{n-1}[\alpha_j x^{(j)}(t_*)+\int_0^{t^*}A_j(s,\ t_*)x^{(j)}(s)ds+ +\int_0^T B_j(s,\ t_*)x^{(j)}(s)ds]=x^*\ ;\ t_*\in(0,T], \tag{4}$$

where $f(t,\ y_0,\ y_1,\ \cdots,\ y_{n-1})$ and its arguments $y_0,\ y_1,\ \cdots,\ y_{n-1}$ are vector functions in Banach space E, together with the given constant vector $x^*\neq 0$. v, T are given positive numbers. Also α_j $(j=0,\ 1,\ \cdots,\ n-1)$ are $n\times n$ constant matrices and $A_j(t,s)$, $B_j(t,s)$ $(j=0,1,\cdots,n\text{-}1)$ are $n\times n$ matrix functions satisfying $B_j\ (t,\ s)\ \big|_{s=0}=0$.

Here if $n=2$ the problem in [7] is obtained, while if $A_j(t, s)=0$ and $B_j(t, s)=0$ $(j=0, 1, \cdots, n-1)$ we can get [1]. As usual, we denote by $C^n([0, T]; E)$ the set of all n-timesdifferentiable and continuous vector functions in $[0, T]$ with values in E. And hence, we can define the first argument of the solution of problem (1)—(4) as the vector function $x(t)$ which belonges to C^n and satisfies conditions (2)—(4).

3. Preliminaries:

(i) —Suppose that $f(t, y_0, y_1, \cdots, y_{n-1})$ is defined and continuous in

$$R: \{[0, T], \ \|y_i\| \leq a_i^* \ (i=0, 1, \cdots, n-1)\}$$

and satisties the following two conditions:

$$\max_{0\leq t\leq T} \|f(t, y_0, y_1, \cdots, y_{n-1})\| \leq M, \tag{5}$$

and

$$\|f(t, x_0, x_1, \cdots, x_{n-1}) - f(t, y_0, \cdots, y_{n-1})\| \leq \sum_{j=0}^{n-1} L_j \|x_j - y_j\| \tag{6}$$

where M, L_j, $j=0, 1, \cdots, n-1)$ are constants.

(ii) Consider the following functions:

$$\phi_q(t) = \int_0^t \frac{(t-s)^{n-q-1}}{(n-q-1)!} f(s)ds, \tag{7}$$

$$y(t) = \sum_{q=0}^{n=1} \Big[\int_0^t A_q(s, t) \int_0^s \frac{(s-s_1)^{n-q-1}}{(n-q-1)!} f(s_1)ds_1 ds +$$

$$+ \int_0^T B_q(s, t) \int_0^S \frac{(s-s_1)^{n-q-1}}{(n-q-1)!} f(s_1)\, ds_1\, ds], \quad 0 \leq s \leq t \leq T, \tag{8}$$

$$P(t) = \sum_{q=0}^{i_0} \Big[\frac{\alpha_{i_0-q}}{q\,!} t^{q-1} + \frac{1}{t} \int_0^t \frac{A_{i_0-q}}{q\,!} s^q ds + \frac{1}{t} \int_0^T \frac{B_{i_0-q}}{q\,!} s^q ds \Big],$$

$$S(t) = P^{-1}(t)\,;\ t \neq 0. \tag{9}$$

$$f(s) = f(s, x(s), x^{(1)}(s), \cdots, x^{(n-1)}(s)),$$

which can be easily shown that they are continous in their intervals of definitions. Also, suppose that they satisfy the conditions:

$$\begin{aligned} &\|\phi_q(t_1) - \phi_q(t_2)\| \leq K_1 |t_1 - t_2|, \\ &\|y(t_1) - y(t_2)\| \leq K_2 |t_1 - t_2|, \end{aligned} \quad ; t_1, t_2 \in [0, T] \tag{10}$$

$$\|S(t_1)-S(t_2)\|\leq K_3|t_1-t_2| \; ; t_1,\ t_2\in (0,\ T]$$

4. Theorem of existence and uniqueness:

Using the following definitions:

$$\delta=\max\|x^*\|,\quad \Delta_q=\max_q\|\alpha_q\|,$$

$$a_q=\max_{0\leq s,t\leq T}\|A_q(s,\ t)\|,\quad \bar{a}_q=\max_{0\leq s,t\leq T}\left\|\frac{\partial A_q(s,\ t)}{\partial_t}\right\|,$$

$$b_q=\max_{0\leq s,t\leq T}\|B_q(s,\ t)\|,\quad \bar{b}_q=\max_{0\leq s,t\leq T}\left\|\frac{\partial B_q(s,\ t)}{\partial t}\right\|,$$

$$C_q=a_q+b_q,\quad \bar{C}_q=\bar{a}_q+\bar{b}_q,$$

$$\Phi_q=\max_{0\leq t\leq T}\|\phi_q(t)\|,\quad Y=\max_{0\leq t\leq T}\|y(t)\|,\quad K=\max_{0<t\leq T}\|S(t)\| \; ; q=0,\ 1,\ \cdots,\ n-1,$$

we can introduce the following existence and uniqueness theorem.

THEOREM: *If conditions* (5) *and* (6) *are satisfied, together with the following conditons*:

$$\mu\equiv K\Big(\delta+\sum_{q=0}^{n-1}\Delta_q\Phi_q+Y\Big)\leq vT, \tag{11}$$

$$\rho_1\equiv\frac{1}{v}\left\{K_3\Big(\delta+\sum_{q=0}^{n-1}\Delta_q\Phi_q+Y\Big)+K\Big(K_2+K_1\sum_{q=0}^{n-1}\Delta_q\Big)\right\}<1, \tag{12}$$

$$\begin{gathered}\frac{T^{i_0-i}}{(i_0-i)!}v+\frac{T^{n-i}}{(n-i)!}M\leq a_i^* \; ;\ i=0,\ \cdots,\ i_0,\\ \frac{T^{n-i}}{(n-i)!}M\leq a_i^* \; ; i=i_0+1,\ \cdots,\ n-1\end{gathered} \tag{13}$$

and

$$\sum_{i=0}^{i_0}L_i\left\{\frac{T^{i_0-i}}{(i_0-i)!}\left[\frac{2vK}{\rho_2 Q}\sum_{q=0}^{n-1}\frac{T^{n-q}}{(n-q)!}\Big(\Delta_q+\frac{C_qT}{(n-q+1)}\Big)\right]+\right.$$
$$\left.+\frac{T^{n-i}}{(n-i)!}\right\}+\sum_{i=i_0+1}^{n-1}\frac{T^{n-i}}{(n-i)!}L_i<1 \tag{14}$$

where

$$\rho_2=\frac{h}{v+K(K_2+K_1\sum_{q=0}^{n-1}\Delta_q)},\quad h=\min_{0<t\leq T}\|S(t)x^*\|,$$

$$Q=v-K\Big(\delta+\sum_{q=0}^{n-1}\Delta_q\Phi_q+Y\Big)-K\Big(K_2+K_1\sum_{q=0}^{n-1}\Delta_q\Big).$$

Then problem (1)—(4) *has a unique solution* $(x(t),\ t_*)$.

5. Some lemmas:

To prove the previous existence and uniqueness theorem we need the following four lemmas.

LEMMA 1: *If functions* $\phi_q(t)$ $(q=0,\ 1,\ \cdots,\ n-1)$, $y(t)$ *and* $S(t)$ *satisfy conditions* (10) *and if conditions* (11), (12) *are satisfied, then the equation*

$$t=\frac{1}{v}\|S(t)\Big\{x^*-\sum_{q=0}^{n-1}\alpha_q\,\Phi_q(t)-y(t)\Big\}\|, \tag{15}$$

has a unique solution $t_*\in(0,\ T]$.

PROOF: Let the operator $F(t)$ denotes the right hand side of equation (15). Therefore

$$0<F(t)\leq\frac{1}{v}\mu$$

and

$$|F(t_1)-F(t_2)|\leq\rho_1|t_1-t_2|.$$

Consequently according to conditions (11), (12) the operator $F(t)$ mapps the interval $(0,\ T]$ into itself and satisfies the condition of contraction, and hence, equation (15) has a unique solution $t_*\in(0,\ T]$.

NOTE: If t_* is the minimal value of t in equation (15) and if $\phi_q(0)=0$ $(q=0,\ 1,\ \cdots,\ n-1)$, $y(0)=0$, then we can estimate that

$$t_*\geq\frac{h}{v+K\Big(K_2+K1\sum_{q=0}^{n-1}\Delta_q\Big)}\equiv\rho_2>0.$$

LEMMA 2: *Let the vector functions* $(\phi_q(t),\ y(t))$ *and* $(\Psi_q(t),\ Z(t))$; $q=0,\ 1,\ \cdots,\ n-1$ *correspond respectively to the roots* t_1 *and* t_2 *of the equations*:

$$\begin{aligned}t&=\frac{1}{v}\|S(t)\{x^*-\sum_{q=0}^{n-1}\alpha_q\,\phi_q(t)-y(t)\}\|,\\ t&=\frac{1}{v}\|S(t)\{x^*-\sum_{q=0}^{n-1}\alpha_q\Psi_q(t)-Z(t)\}\|,\end{aligned} \tag{16}$$

and satisfy conditions (10)—(12). *Also if* η_1, η_2 *are two vector functions of the form*:

$$\eta_1=\frac{S(t_1)}{t_1}\{x^*-\sum_{q=0}^{n-1}\alpha_q\,\phi_q(t_1)-y(t_1)\},$$

$$\eta_2=\frac{S(t_2)}{t_2}\{x^*-\sum_{q=0}^{n-1}\alpha_q\ \Psi_q(t_2)-Z(t_2)\}\},\tag{17}$$

then we can estimate that

$$|t_1-t_2|\leq\frac{KW}{Q}\ \ and\ \|\eta_1-n_2\|\leq\frac{(2vK)W}{\rho_2 Q}\tag{18}$$

$$where\ W=\sum_{q=0}^{n-1}\triangle_q\max_{0\leq t\leq T}\|\ \phi_q(t)-\Psi_q(t)\|+\max_{0\leq t\leq T}\|\ y(t)-Z(t)\|\tag{19}$$

and $K, Q,\ \rho_2$ *are defined as before.*

PROOF: Since t_1, t_2 are the roots of equations (16), then

$$|t_1-t_2|\leq\frac{1}{v}\|S(t_1)\{x^*-\sum_{q=0}^{n-1}\alpha_q\ \phi_q(t_1)-y(t_1)\}$$
$$-S(t_2)\{x^*-\sum_{q=0}^{n-1}\alpha_q\ \Psi_q(t_2)-Z(t_2)\}\|,$$

which from (10) gives

$$v|t_1-t_2|\leq[k_3(\delta+\sum_{q=0}^{n-1}\triangle_q\ \Phi_q+Y)+k(k_2+k_1\sum_{q=0}^{n-1}\triangle_q)]\ |t_1-t_2|+k(\sum_{q=0}^{n-1}\triangle_q$$
$$\times\max_{0\leq t\leq T}\|\phi_q(t)-\Psi_q(t)\|+\max_{0\leq t\leq T}\|y(t)-Z(t)\|).$$

Therefore, using (19) the last inequality leads to the first inequality of (18). Also form (17), using the well known inequality

$$\|\frac{A}{\|A\|}-\frac{B}{\|B\|}\|\leq\frac{2\|A-B\|}{\max\{\|A\|,\ \|B\|\}}$$

we can obtain

$$\|\eta_1-\eta_2\|\leq\frac{2}{\gamma}\|(S(t_1)-S(t_2))x^*-\sum_{q=0}^{n-1}\alpha_q(S(t_1)\ \phi_q(t_1)$$
$$-S(t_2)\ \Psi_q(t_2))-S(t_1)\ y(t_1)-S(t_2)\ Z(t_2))\|$$

where

$$\gamma=\max\{\|S(t_1)\ \ [x^*-\sum_{q=0}^{n-1}\alpha_q\ \phi_q(t_1)-y(t_1)\|,$$
$$\|S(t_2)\ \ [x^*-\sum_{q=0}^{n-1}\alpha_q\ \Psi_q(t_2)-Z(t_2)]\|\}.$$

Using (10) and (19) we can have

$$\|\eta_1-\eta_2\|\leq\frac{2[(v-Q)|t_1-t_2|+KW]}{\rho_2}$$

and hence, substituting from the first inequality of (18), the second inequality of (18) will be hold.

LEMMA 3. *If the vector functions $\phi_q(t)(q=0, 1, \cdots, n-1)$ and $y(t)$ are defined as in (7), (8), then the following is true:*

$$\max_{0\le t\le T}\|\phi_q(t)\|\le M\frac{T^{n-q}}{(n-q)!}, \tag{20}$$

$$\max_{0\le t\le T}\|\phi_q'(t)\|\le M\frac{T^{n-q-1}}{(n-q-1)!}, \tag{21}$$

$$\max_{0\le t\le T}\|y(t)\|\le M\sum_{q=0}^{n-1}\frac{T^{n-q+1}}{(n-q+1)!}C_q, \tag{22}$$

$$\max_{0\le t\le T}\|y'(t)\|\le M\sum_{q=0}^{n-1}\frac{T^{n-q}}{(n-q)!}\left(a_q+\frac{\overline{C}_q T}{n-q+1}\right), \tag{23}$$

where M, a_q, C_q, $\overline{C}_q$ are defined as before.

LEMMA 4: *Let $P(t)$ be the non-singular matrix function which is defined in (9), then for its inverse matrix $S(t)$ we have*

$$\max_{0<t\le T}\|S(t)\|\le\frac{n^2}{N}(n-1)^{\frac{n-1}{2}}\left[\sum_{q=0}^{i}\frac{T^{q-1}}{q!}\left(\Delta_{i_0-q}+\frac{C_{i_0-q}T}{q+1}\right)\right]^{(n-1)}, \tag{24}$$

$$\max_{0<t\le T}\|S'(t)\|\le\frac{n^4}{N^2}(n-1)^{n-1}\left[\sum_{q=0}^{i_0}\frac{T^{q-1}}{q!}\left(\Delta_{i_0-q}+\frac{C_{i_0-q}T}{q+1}\right)\right]^{2(n-1)}\times$$

$$\times\sum_{q=0}^{i_0}\frac{T^{q-2}}{(q-1)!}\left[\Delta_{i_0-q}+a_{i_0-q}\frac{T}{q}+(C_{i_0-q}+\overline{C}_{i_0-q})\frac{T}{q(q+1)}\right], \tag{25}$$

where $N=\min_{0<t\le T}|\det P(t)|$.

PROOF: The inequality (24) is easily followed by using Adamar's inequailty [3]. Also, since

$$S'(t)=-S(t)P'(t)S(t),$$

then the inequality (25) can be easily obtained.

6. Proof of the theorem of Existence and uniqueness:

Let $U\subset C^n$, where

$$U:\{[0,\ T],\ \|x^{(i)}(t)\|\le a_i^*\ (i=0, 1, \cdots, n-1)\}.$$

Suppose that $x(t)\in U$

It is easily to see that problem (1)−(4) is equivalent to the system

$$x(t)=\frac{\eta_t i_0}{i_0\,!}+\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}\,f(s)ds, \tag{26-1}$$

$$\|\eta\|=v=\frac{1}{t_*}\|S(t_*)\,[x^*-\sum_{q=0}^{n-1}\alpha_q\,\phi_q(t_*)-y(t_*)]\,\|, \tag{26-2}$$

where $\phi_q(t_*)$ $(q=0,\ 1,\ \cdots,\ n-1)$, $y(t_*)$, $S(t_*)$ are defined as in (7), (8), (9) respectively.

According to conditions (11), (12) for every element $x(t)\in U$ equation (26-2) has a unique solution $t_*(x)$.

Now we define in U the operator D such that

$$D\times(t)=\frac{t^{i_0}\eta(x)}{i_0\,!}+\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}\,f(s)ds,$$

$$\text{where } \eta(x)=\frac{S(t^*(x))}{t_*(x)}[x^*-\sum_{q=0}^{n-1}\alpha_q\ \phi_q(t_*(x))-y(t_*(x))].$$

Therefore,

$$\|D^{(i)}x(t)\|\le\frac{T^{i_0-i}}{(i_0-i)!}v+M\frac{T^{n-i}}{(n-i)!}\ ;\ i=0,\ 1,\ \cdots,\ i_0$$

and

$$\|D^{(i)}\ x(t)\|\le M\frac{T^{n-i}}{(n-i)!}\ ;\ i=i_0+1,\ \cdots,\ n-1$$

and hence from (13) it follows that $D^{(i)}\,x(t)\in U$ $(i=0,\ 1,\ \cdots,\ n-1)$ i.e. the operator D mapps the set U into itself.

Let $x_1(t)$, $x_2(t)\in U$, then

$$\|D^{(i)}x_1(t)-D^{(i)}x_2(t)\|\le\|\eta(x_1)-\eta(x_2)\|\frac{T^{i_0-i}}{(i_0-i)!}+$$

$$+\frac{T^{n-i}}{(n-i)!}\sum_{j=0}^{n-1}L_j\|x_1^{(j)}(t)-x_2^{(j)}(t)\|\ ;\ i=0,\ 1,\ \cdots,\ i_0$$

and

$$\|D^{(i)}x_1(t)-D^{(i)}x_2(t)\|\le\frac{T^{n-i}}{(n-i)!}\sum_{j=0}^{n-1}L_j\|x_1^{(j)}(t)-x_2^{(j)}(t)\|\ ;\ i=i_0+1,\ \cdots,\ n-1$$

Using Lemmas (2)−(4) we can obtain

$$\max_{0\le t\le T}\|D^{(i)}x_1(t)-D^{(i)}x_2(t)\|\le\Big[\frac{2vk}{\rho_2 Q}\Big(\sum_{q=0}^{n-1}\frac{T^{n-q}}{(n-q)!}\Big(\Delta_q+$$

$$+\frac{C_qT}{(n-q-1)}\Big)\Big)\frac{T^{i_0-i}}{(i_0-i)!}+\frac{T^{n-i}}{(n-i)!}\Big]\sum_{j=0}^{n-1}L_j\max_{0\le t\le T}\|x_1^{(j)}(t)-x_2^{(j)}(t)\|$$

$$;\ i=0,\ 1,\ \cdots,\ i_0 \qquad (27\text{-}1)$$

and

$$\max_{0\le t\le T}\|D^{(i)}x_1(t)-D^{(i)}x_2(t)\|\le\frac{T^{n-i}}{(n-i)!}\sum_{j=0}^{n-1}\max_{0\le t\le T}\|x_1^{(j)}(t)-x_2^{(j)}(t)\|;$$

$$;\ i=i_0+1,\ \cdots,\ n-1 \qquad (27\text{-}2)$$

Introducing the generalized norm [5] by the equality

$$|x|=\begin{vmatrix}\max\|x\|\\ \max\|x^{(1)}\|\\ \vdots\\ \max\|x^{(n-1)}\|\end{vmatrix}.$$

The inequalities in (27-1), (27-2) can be put in the form

$$|Dx_1-Dx_2|\le S^*|x_1-x_2|,$$

where

$$S^*=\begin{vmatrix}u_0L_0 & u_0L_1 & \cdots\cdots & u_0L_{n-1}\\ u_1L_0 & u_1L_1 & \cdots\cdots & u_1L_{n-1}\\ \vdots & & & \\ u_{n-1}L_0 & u_{n-1}L_1 & \cdots\cdots & u_{n-1}L_{n-1}\end{vmatrix},$$

$$u_i=\frac{T^{i_0-i}}{(i_0-i)!}\Big[\frac{2vk}{\rho_2Q}\Big(\sum_{q=0}^{n-1}\frac{T^{n-q}}{(n-q)!}\Big(\Delta_q+\frac{C_qT}{(n-q+1)}\Big)\Big)\Big]$$

$$+\frac{T^{n-i}}{(n-i)!};\qquad i=0,\ 1,\ \cdots,\ i_0$$

and

$$u_i=\frac{T^{n-i}}{(n-i)!};\qquad i=i_0+1,\ \cdots,\ n-1.$$

It is easily to verify that S^* is the a-matrix [5] if and only if (14) is satisfied. Conseqently, according to the generalization of principle of contraction mapping [5], we can say that the operator D has in U a unique fixed point $x(t)$.

From above, it follows that problem (1)−(4) has a unique solution $(x(t),\ t^*)$.

This completes the proof of the theorem.

REMARK: Again problem (1)−(4) will be considered later to study the stability of its solution, whose existence and uniqueness have been proved here.

Dept. of Math, Faculty of Science,
Al-Azhar University, Nasr City,
Cairo, Egypt.

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