Kyungpook Math. J.
Volume 23, NLmber 2
December, 1983

GENERALIZED ASCENDING CHAINS AND FORMAL POWER SERIES RINGS

By Lovis Dale

1. Introduction

Generalized ascending chains of ideals arise naturally in polynomial rings in several variables. By using such ideals we were able to prove the Hilbert basis theorem for a polynomial ring in several variables without using the usual method of extending variables. In this paper we will prove a similar result for a formal power series ring in several variables. This will be done by using a modification of the argument used to prove the theorem for a polynomial ring in several variables.

Let P be the set of non-negative integers and P^n the product of P n-times. If $\alpha=(i_1, \cdots, i_n)$, $\beta=(j_1, \cdots, j_n)\in P^n$. then we will say that

(i) $\alpha=\beta$ if $i_k=j_k$ for each $k\in\{1, 2, \cdots, n\}$; and

(ii) $\alpha<\beta$ if $i_k<j_k$ for some $k\in\{1, 2, \cdots, n\}$ and

$i_t\leq j_t$ for all $t\neq k$.

If either $\alpha=\beta$ or $\alpha<\beta$, then we will say that $\alpha\leq\beta$. We define the sun of α and β to be $\alpha+\beta=(i_1+j_1, \cdots, i_n+j_n)$.

1.1. DEFINITION. A collection of ideals $\{A_\alpha|\alpha\in P^n\}$ in a ring R will be called a *generalized ascending chain of dimension n* if whenever A_α and A_β are two ideals in the collection with $\alpha\leq\beta$, then $A_\alpha\subseteq A_\beta$. The generalized ascending chain of ideals $\{A_\alpha|\alpha\in P^n\}$ is called *finite* if there is a $\triangle=(N, N, \cdots, N)\in P^n$ such that for each $\alpha=(i_1, i_2, \cdots, i_n)\in P^n$ with $\max_\kappa i_k\geq N$ we have $A_\alpha=A_\beta$ for some $\beta\leq\triangle$. If we consider the elements of P^n to be lattice points, then to say that $\{A_\alpha|\alpha\in P^n\}$ is *finite* means that there is an n-dimensional cube C_N of length N such that for any $\alpha\in P_n$ and $\alpha\notin C_N$ there is a $\beta\in C_N$ with $A_\alpha=A_\beta$.

It was proved in [1] that if R is a Noetherian ring, then every generalized chain of ideals in R is finite. If R_1, R_1, $\cdots$, R_n are rings, then generalized ascending chains arise naturally in the direct sum $R=R_1\oplus R_2\oplus\cdots\oplus R_n$. It is easy to show that every generalized ascending chain in R is finite if and only if every generalized ascending chain in each R_i is finite.

2. Power series rings in several variables.

Let R be a commutative ring with an identity and $R[[x_1, x_2, \cdots, x_n]]$ the formal power series ring in the indeterminates $x_1, x_2, \cdots, x_n$. An element in $R[[x_1, x_2, \cdots, x_n]]$ is of the form $f=\Sigma a_{i_1\cdots i_n} x_1^{i_1}\cdots x_n^{i_n}$ with each i_k being unrestricted. The coefficients of f are also unrestricted. Let $\alpha=(i_1, i_2, \cdots, i_n)$, $|\alpha|=i_1+i_2+\cdots+i_n$ and write $a_\alpha=a_{i_1\cdots i_n}$ and $X^\alpha=x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$. Then the polynomial f may be written $f=\Sigma a_\alpha X^\alpha$ with $|\alpha|\to\infty$. The degree of a non-zero term $a_\alpha X^\alpha$ of f is $|\alpha|$. Now for each α with $|\alpha|=m$, f may contain more than one term of degree m. If f_m is the sum of all terms of f of degree m, then clearly f_m is a homogeneous polynomial. Consequently, we can write $f=\sum_0^\infty f_m$, where f_m is a homogeneous polynomial of degree m. Thus each non-zero polynomial f contains a homogeneous polynomial of lowest degree. For each polyonmial f_m we consider the lexicographic ordering for the terms of f_m, i.e., if $a_\alpha X^\alpha$ and $a_\tau X^\tau$ are terms of f_m with $\alpha=(i_1, i_2, \cdots, i_n)$ and $\tau=(t_1, t_2, \cdots, t_n)$such that $i_1=t_1$, $i_2=t_2$, $\cdots$, $i_s=t_s$ but $i_{s+1}>t_{s+1}$ $(s\geq 0)$, then we say that $a_\alpha X^\alpha$ is higher than $a_\tau X^\tau$ or $a_\tau X^\tau$ is lower than $a_\alpha X^\alpha$. It is clear that f_m can have only one lowest term. Consequently, the polynomial f can have only one lowest term of lowest degree. We will call such a term the *lowest term* of f. Now if H is an ideal in $R[[x_1, x_2, \cdots, x_n]]$, let

$H_\alpha=\{b\in R|b=0$ or bX^α is the lowest term of some $f\in H\}$.

2.1. LEMMA. *If H is an ideal in $R[[x_1, x_2, \cdots, x_n]]$, then $\{H_\alpha|\alpha\in P^n\}$ is a generalized ascending chain of ideals in R.*

PROOF. If $a, b\in H_\alpha$ and $r\in R$, then $a-b$ and ra are elements in H_α as one sees by taking the difference of the corresponding polynomials and r times the corresponding polynomial. Consequently, each H_α is an ideal. Now let $0\neq b\in H_\alpha$ and $\beta\in P^n$ such that $\alpha<\beta$. If $\alpha=(i_1, i_2, \cdots, i_n)$ and $\beta=(j_1, j_2, \cdots, j_n)$ then $j_k=i_k+t_k$ for some $t_k\geq 0$. Let $\tau=(t_1, t_2, \cdots, t_n)$ and f be a polynomial in H with bx^α as lowest term. Then $X^\tau bX^\alpha=bX^{\tau+\alpha}=bX^\beta$ is the lowest term of the polynomial xf. Consequently, $b \in H_\beta$ and it follows that $H_\alpha \subseteq H_\beta$. Therefore $\{H_\alpha | \alpha \in P^n\}$ is a generalized ascending chain of ideals in R. The ideals $\{H_\alpha | \alpha \in P^n\}$ will be called the *lowest coefficient ideals* of H.

2.2. THEOREM. *If R is a Noetherian ring, then $R[[x_1, x_2, \cdots, x_n]]$ is also Noetherian.*

PROOF. Let H be an ideal in $R[[x_1, x_2, \cdots, x_n]]$ and $\{H_\alpha | \alpha \in P^n\}$ the corresponding lowest coefficient ideals. It follows from Lemma 2.1 that this collection of ideals is a generalized ascending chain of ideals in R and since R is Noetherian this collection is finite, i.e., there exists $\triangle=(N, N, \cdots, N)$ such that for each $\alpha=(i_1, i_2, \cdots, i_n) \in P^n$ with $\max_k i_k \geq N$, $H_\alpha = H_\beta$ for some $\beta \leq \triangle$. Since R is Noetherian, each H_α is finitely generated, say by elements $b_{\alpha 1}$, $b_{\alpha 2}$, $\cdots$, $b_{\alpha m_\alpha}$. By the Axiom of Choice, for each $\alpha \leq \triangle$ and $k \in \{1, 2, \cdots, m_\alpha\}$ we can choose a polynomial $f_{\alpha k}$ in H with $b_{\alpha k}$ as the coefficient of its lowest term. The proof of the theorem will be completed by showing that the finite set $\{f_{\alpha k} : 1 \leq k \leq m_\alpha, \alpha \leq \triangle\}$ generates H. To this end, consider a typical polynomial $f = \Sigma a_\lambda X^\lambda$ in H with lowest term $a_\tau X^\tau$. Let $|\tau| = r$. Then the least degree of f is r. If $\tau = (t_1, t_2, \cdots, t_n)$, then $t_j > N$ for some j or $t_j \leq N$ for each j. If $t_j > N$ for some j, then there exists $\alpha \leq \triangle$ such that $\tau > \alpha$ and $H_\tau = H_\alpha$. Hence the lowest coefficients of the polonomials

$$X^{\tau-\alpha} f_{\alpha 1},\ X^{\tau-\alpha} f_{\alpha 2},\ \cdots,\ X^{\tau-\alpha} f_{\alpha m},$$

generate H_τ. Thus there are elements $c_{\tau 1}$, $c_{\tau 2}$, $\cdots$, $c_{\tau m_\tau}$ in R such that the lowest term of $f_1 = f - \Sigma c_{\tau k} f_{\alpha k}$ is higher than that of f or the least degree of f_1 is higher than r, and f_1 lies in H. If f and f_1 have the same lowest degree r, then we can repeat the process with f_1. Since there are only a finite number of terms of f of degree r that are higher than a given one, a finite number of applications of this process will yield a polynomial $g = f - \Sigma c_{\tau k} f_{\alpha k}$, the sum taken over all τ with $|\tau| = r$ and corresponding α, such that the least degree of g is greater than r and g lies in H. On the other hand, if $t_j \leq N$ for all j, then $\tau \leq \eta$ and a process similar to the above will yield a polynomial $g' = f - \Sigma c_{k\tau} f_{\tau k} \in H$ such that the least degree of g' is greater than r. Consequently, by induction on the least degree of f we can find a polynomial $h \in H$ generated by $\{f_{\alpha_k}\}$, $\alpha \leq \triangle$, such that $f - h = 0$. Therefore H is generated by $\{f_{\alpha_k}\}$, $\alpha \leq \triangle$.

and it follows that $R[[x_1, x_2, \cdots, x_n]]$ is Noetherian.

University of Alabama in Birmingham
Birmingham, Alabama 35294

REFERENCES

[1] L. Dale, ***Generalized Ascending Chains with an Application to Hibert's Theorem,*** Journal of Natural Sciences and Mathematics 19 #2(1979), 231—238.

[2] D. Hilbert, ***Gesammelte Abhandlungen,*** Volume 1, Chelsea, New York, 1965.