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UNIQUE DETERMINATION OF ANY ANALYTIC FUNCTION OF TWO REAL VARIABLES FROM ITS VALUES GIVEN ON THE POINTS OF A DENUMERABLE SET

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This paper is in the setup of real numbers. Let $f(x, y)$ be an analytic function (of two real variables x and y) in a nonempty open disk D with center at the origin (0, 0). As such, $f(x, y)$ has a power series expansion in x and y valid in D given by:

$$(1)\qquad f(x, y)=a_{00}+(a_{10}x+a_{01}y)+(a_{20}x^2+a_{11}xy+a_{02}y^2)+(a_{30}x^3+\cdots)+$$
$$+\cdots+(a_{h0}x^h+a_{h-1,1}x^{h-1}y+\cdots+a_{mn}x^my^n+\cdots+a_{0h}y^h)+\cdots$$

For our convenience, we have written $f(x, y)$ in (1) as a sum of infinitely many homogeneous polynomials $P_h(x, y)$ of degree h with $h=0, 1, 2, \cdots$ where

$$(2)\qquad P_h(x, y)=a_{h0}x^h+a_{h-1,1}x^{h-1}y+\cdots+a_{mn}x^my^n+\cdots+a_{1,h-1}xy^{h-1}+a_{0h}y^h$$

Let g be a function of a complex variable z such that g is analytic in an open disk $|z|<r$. We recall [1, p. 87] that g is uniquely determined in D by its values given on the points of any denumerable subset S of $|z|<r$ such that 0 is a limit point of S. This is not the case in connetion with real analytic functions. For instance, the function xy as well as x^2y vanishes at every point of the denumerable set $\{(0, k^{-1})|k=1, 2, 3, \cdots\}$ and yet, xy and x^2y are not identical in any nonempty open isk D (of the xy-plane) with center at the origin (0, 0). However, as shown below, there always exists a denumerable subset E of D such that if two real analytic functions agree on E then they are identical.

In what follows, we let $(p_k)_{k=0,1,2,\cdots}$ be a sequence of nonzero real numbers which converge to 0. Thus,

$$(3)\qquad \lim_{k\to\infty} p_k=0 \text{ with } p_k\neq 0 \text{ for } k=0, 1, 2, \cdots$$

Also, in what follows, we let E be the denumerable subset of the xy-plane

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defined by:

$$(4) \qquad E=\{(p_k^{n+1},\ p_k^{n+2})\,|\,k,\ n=0,\ 1,\ 2,\ \cdots\}$$

where p_k is given by (3). From (3) it follows that

$$(5) \qquad \lim_{k\to\infty}(p_k^{n+1},\ p_k^{n+2})=(0,0) \text{ for } n=0,\ 1,\ 2,\ \cdots$$

Moreover, we let D be a nonempty open disk (of the xy-plane) with center at $(0,0)$. In view of (3), we may assume (without loss of generality) that E is *a* subset of D.

Furthermore, as mentioned above, we let $f(x, y)$ be an analytic function (of two real variables x and y) defined in D whose power series expansion in D is given by (1).

Finally, let the following real numbers

$$(6) \qquad f(p_0^1,\ p_0^2),\ f(p_1^1,\ p_1^2),\ f(p_2^1,\ p_2^2),\ \cdots,\ f(p_0^2,\ p_0^3),\ f(p_1^2,\ p_1^3),\ f(p_2^2,\ p_2^3),\ \cdots,$$
$$f(p_0^{n+1},\ p_0^{n+2}),\ f(p_1^{n+1},\ p_1^{n+2}),\ f(p_2^{n+1},\ p_2^{n+2}),\ \cdots,\ f(p_k^{n+1},\ p_k^{n+2}),\ \cdots$$

be given which represent the values of $f(x, y)$ at the points of the subset E of D where E is as given in (4).

Now, based on (3), (4) and (5), we determine (uniquely) the values of a_{mn} 's in (1), which in turn determine uniquely $f(x, y)$ in the entire D.

To determine a_{00} let us take from both sides of equality (1) limit

$$(7) \qquad \text{as } k\to\infty \text{ with } (x,\ y)=(p_k,\ p_k^2)$$

Since $f(x, y)$ is analytic (and a fortiori continuous) in D, clearly $\lim_{k\to\infty} f(x,y)$ is uniquely determined (in fact is equal to $f(0, 0)$) by its values $f(p_0,\ p_0^2)$, $f(p_1,\ p_1^2)$, $f(p_2,\ p_2^2)$, $\cdots$ which are given in (6). Also, in view of (5), it follows readily that the limit (according to (7)) of the series immediately to the right of a_{00} in (1) is equal to 0. Hence,

$$(8) \qquad a_{00}=\lim_{k\to\infty} f(p_k,\ p_k^2)$$

and therefore a_{00} is uniquely determined by (6).

To determine a_{10} let us subtract a_{00} from both sides of equality (1) and then divide both sides by x and then take from both sides limit according to (7). From (5) it readily follows that the limit (according to (7)) of the product of x^{-1} and the series immediately to the right of $a_{10}x$ in (1) is equal to 0. This is because the limit according to (7) of each of $y/x,\ x,\ y,\ y^2/x,\ \cdots$ is equal to 0. Hence (using (8)),

$$(9) \qquad a_{10} = \lim_{k\to\infty} \frac{f(p_k,\ p_k^2) - a_0}{p_k}$$

and therefore a_{10} is uniquely determined by (6).

To determine a_{01} let us subtract $a_{00} + a_{10}x$ from both sides of equality (1) and then divide both sides by y and then take from both sides limit

$$(10) \qquad \text{as } k\to\infty \text{ with } (x,\ y) = (p_k^2,\ p_k^3)$$

From (5) it readily follows that the limit (according to (10)) of the product of y^{-1} and the series immediately to the right of $a_{01}y$ in (1) is equal to 0. This is because the limit according to (10) of each of $x^2/y,\ x,\ y,\ x^3/y,\ \cdots$ is equal to 0. Hence (using (8) and (9)),

$$(11) \qquad a_{01} = \lim_{k\to\infty} \frac{f(p_k^2,\ p_k^3) - a_{00} - a_{10}\,p_k^2}{p_k^3}$$

and therefore a_{01} is uniquely determined by (6).

To clarify our procedure we explicitly calculate two more coefficients.

To determine a_{20} let us subtract $a_{00} + a_{10}\,x + a_{01}\,y$ from both sides of equality (1) and then divide both sides by x^2 and then take from both sides limit according to (7). From (5) it readily follows that the limit (according to (7)) of the product of x^{-2} and the series immediately to the right of $a_{20}x^2$ in (1) is equal to 0. This is because the limit according to (7) of each of $y/x,\ x,\ y,\ y^2/x,\ y^3/x,\ \cdots$ is equal to 0. Hence (using (8), (9) and (11)),

$$(12) \qquad a_{20} = \lim_{k\to\infty} \frac{f(p_k,\ p_k^2) - a_0 - a_{10}p_k - a_{01}p_k^2}{p_k^2}$$

and therefore a_{20} is uniquely determined by (6).

To determine a_{11} let us subtract $a_{00} + a_{10}x + a_{01}y + a_{20}x^2$ from both sides of equality (1) and then divide both sides by xy and then take from both sides limit according to (10). From (5) it readily follows that the limit (according to (10)) of the product of $x^{-1}y^{-1}$ and the series immediately to the right of $a_{11}xy$ in (1) is equal to 0. This is because the limit according to (10) of each of $y/x,\ x^2/y,\ x,\ y,\ y^2/x,\ \cdots$ is equal to 0. Hence (using (8), (9), (11) and

(12)),

(13) $$a_{11}=\lim_{k\to\infty}\frac{f(p_k^2,\ p_k^3)-a_0-a_{10}p_k^2-a_{01}p_k^3-a_{20}p_k^4}{p_k^5}$$

and therefore a_{11} is uniquely determined by (6).

From (8), (9), (11), (12) and (13) we see that each of the coefficients a_{00}, a_{10}, a_{01}, a_{20}, a_{11} is obtained in terms of the previous ones. Moreover, a_{00}, a_{10}, a_{20} are obtained by taking limit according to (7), whereas c_{01}, a_{11} are obtained by taking limit according to (10).

We claim that in general a_{mn} appearing in (1) is uniquely determined in terms of a_{00}, a_{10}, a_{20}, a_{11}, $\cdots$, $a_{m+1,n-1}$. Moreover, after performing the required subtraction and division for m, $n=0, 1, 2, \cdots$

(14) a_{mn} <u>is obtained by taking limit as</u> $k\to\infty$ <u>with</u> $(x, y)=(p_k^{n+1},\ p_k^{n+2})$

We note that in (14) it is the case that $(p_k^{n+1},\ p_k^{n+2})$ is independent of m. This is in accordance with the fact that a_{00}, a_{10}, a_{20} are obtained by taking limit according to (7), whereas a_{01}, a_{11} according to (10).

To prove our claim, let us suppose that a_{00}, a_{10}, a_{01}, a_{20}, $\cdots$, $a_{m+1,n-1}$ are determined. Next, let us subtract $a_{00}+a_{10}x+a_{01}y+a_{20}x^2+\cdots+a_{m+1,n-1}x^{m+1}y^{n-1}$ from both sides of equality (1) and then divide both sides by $x^m y^n$ and then take from both sides limit according to (14). From (5) it readily follows that the limit (according to (14)) of the product $x^{-m}y^{-n}$ and the series immediately to the right of $a_{mn}x^m y^n$ in (1) is equal to 0. This is because the limit according to (14) of each of

(15) $$\frac{y}{x},\ \left(\frac{y}{x}\right)^2,\ \cdots,\ \left(\frac{y}{x}\right)^m;\ \frac{x^{n+1}}{y^n},\ \frac{x^n}{y^{n-1}},\ \cdots,\ \frac{y^{m+1}}{x^m};\ \frac{x^{n+2}}{y^n},\ \cdots$$

is equal to 0. Hence,

(16) $$a_{mn}=\lim_{k\to\infty}\frac{f(p_k^{n+1},\ p_k^{n+2})-a_0-a_{10}p_k^{n+1}-a_{01}p_k^{n+2}-\cdots-a_{m+1,n-1}p_k^{(m+1)(n+1)+(n-1)(n+2)}}{p_k^{m(n+1)+n(n+2)}}$$

and therefore a_{mn} is uniquely determined by (6).

The reason that in (14) we have chosen $(x, y)=(p_k^{n+1},\ p_k^{n+2})$ is precisely to make the limit according to (14) of the two essential ratios y/x and x^{n+1}/y^{m}

in (15) equal to 0.

Obviously, in view of (1), (3), (4), (6) and (16) we have proved:

THEOREM. *Let $f(x, y)$ be an analytic function of two real variables x and y in a nonempty open disk D with center at* (0, 0). *Then $f(x, y)$ is uniquely determined by its values at the points of a denumerable subset* $\{(p_k^{n+1}, p_k^{n+2}) \mid k, n=0, 1, 2, \cdots\}$ *of D where $(p_k)_{k=0,1,2,\cdots}$ is a sequence of nonzero real numbers which converges to* 0.

It is clear how to modify the statement of the Theorem when it refers to a disk with center other than (0,0) or to real analytic functions of more than two variables.

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REFERENCE

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