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ON THE PROJECTIVE HOMOLOGY GROUPS

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In [1] it is proposed the idea of the projective homology groups of compact spaces, which is a generalization of the interesting homology groups of Steenrod, [9]. These groups depend on a new type of cycle. In essence it is an infinite cycle (in a compace space X) with the (regularity) requirement that the successive simplexes are contained in any finite open covering of the space X, [4]. In [1] and [2] it is proved that the projective homology groups satisfies the first five axioms of the seven axioms of Eilenberg-Steenrod, [5].

The main objective of the present papor is to present a proof of the excision axiom (The sixth one of Eilenberg-Steenrod's axioms) for the projective homology groups defined over a chain complex as a coefficient group.

1. Projective homology groups over a chain complex

Let $\mathscr{K}$ be the category of pairs of countably locally finite simplicial complexes and their inclusion maps, Q the category of pairs of compact spaces and their continuous maps, and $G=\{G^i, d^i\}$ be a chain complex of commutative groups, where $d^i : G^i \longrightarrow G^{i+1}$ for each integer i. For every object (X, A) of Q consider the set $w(X, A)$ of the triples $\alpha=(K, L:f)$, where (K, L) is an object of the category $\mathscr{K}$ and $f:(K, L)\longrightarrow(X, A)$ is a regular map, i.e., for every finite open covering O of the space X amost f-images of the simplexes of K are contained in the members of O. If $\beta=(K_1, L_1:f_1)\in w(X, A)$ then we consider $\alpha<\beta$ if in K there exists

$\pi_{\alpha\beta}:(K, L)\longrightarrow(K_1, L_1)$ such that $f_1\,\pi_{\alpha\beta}=f$, [4] : and we say $\alpha<\beta$ in virtue of $\pi_{\alpha\beta}$. Let $C^{\alpha,i}_{n+i}$ be the group $C_{n+i}(K, L:G^i)$ of the chains of (K, L) over the coefficient group G^i, and $D^\alpha_n=\prod_i C^{\alpha,i}_{n+i}$. The boundry map $\triangle^\alpha_n : D^\alpha_n\longrightarrow D^\alpha_{n-1}$ is defined as shown in the following diagram:

$$\begin{array}{lccccccc}
D_n^\alpha : \cdots & + & C_{n+i-1}^{\alpha,i-1} & + & C_{n+i}^{\alpha,i} & + & C_{n+i+1}^{\alpha,i+1} + \cdots \\
\Big\downarrow \overset{\alpha}{\Delta}_n & & \Big\downarrow \partial_\alpha^i \searrow d_\alpha^i & & \Big\downarrow \partial_\alpha^i \searrow d_\alpha^{i-1} & & \\
D_{n-1}^\alpha : \cdots & + & C_{n+i-2}^{\alpha,i-1} & + & C_{n+i-1}^{\alpha,i} & + & C_{n+i}^{\alpha,i+1} + \cdots,
\end{array}$$

i.e., if $X_n \in D_n^\alpha$ and its i^{th} coordinate is $X^i \in C_{n+i}^{\alpha, i}$, then $\Delta_n^\alpha X^i = \partial_\alpha^i X^i + (-1)^{n+i} d_\alpha^i X^i$, where ∂_α^i is a boundary homomorphism and d_α^i is the homomorphism induced by d^i. The homology groups of the chain complex $\{D_n^\alpha, \Delta_n^\alpha\}$ is denoted by $\{H_n^\alpha\}$. If $\alpha < \beta$ then $\pi_{\alpha\beta}$ defines, consequently, the homomorphisms $\pi_{\alpha\beta} : D_n^\alpha \longrightarrow D_n^\beta$ and $\pi_{\alpha\beta *} : H_n^\alpha \longrightarrow H_n^\beta$, [6].

DEFINITION 1. *The projective homology group* of a pair (X, A) over the chain complex G is the limit of the direct system $\{H_n^\alpha, \pi_{\alpha\beta *}\}$ over the directed set $w(X, A)$; it is denoted by $H_n(X, A : G)$; or, breifly H_n, i.e., $H_n(X, A : G) = H_n = \underset{\overrightarrow{\alpha}}{\mathrm{Lim}} \{H_n^\alpha, \pi_{\alpha\beta *}\}$.

If G is a trivial chain complex, [6], the H_n is isomorphic to the projective homology group defined in [2].

If $g : (X, A) \longrightarrow (Y, B)$ is in the category Q and $\alpha = (K, L : f) \in w(X, A)$ then $g(\alpha) = (K, L : gf)$ is in the set $w(Y, B)$. The induced homomorphism $g_* : H_n(X, A : G) \longrightarrow H_n(Y, B : G)$ is defined as follows: if $A_n \in H_n(X, A : G)$ with representative $h \in H_n^\alpha$ then $h \in H_n^{g(\alpha)}$ and it is a representative of $g_*(A_n)$.

If the boundary homomorphism ∂_* of the groups $\{H_n\}$ (compare with [3]) then it is easy to show that the triple $\mathscr{H} = \{H_n, g_*, \partial_*\}$, which is called the projective homology construction over the chain complex G, is naturally equivalent to the construction H defined in [3]. Moreover, we can prove that the triple $\mathscr{H}$ satisfies the first five axioms of Eilenberg-Steenrod list of axioms, [3].

2. The excision axiom

The object of this section is to verify the sixth one of the Eilenberg-Steenrod's axioms for the construction $\mathscr{H}_n$. The proof depends on the following notions.

DEFINITION 2. If $(X, A) \in Q$ and $\alpha=(K;f)$ is in $w(X)$, then an element x^i of the group $C^{\alpha,i}_{n+i}=C_{n+i}(K;G^i)$ is said to be a *regular chain* of X relative to A if, for each neighborhood V of A, $f(t) \subset V$ for almost all simplexes t of the carrier $|x^i|$ of x^i.

The subset $\overline{C}^{\alpha,i}_{n+i}$ of $C^{\alpha,i}_{n+i}$ consisting of the regular chains of X relative to A is a subgroup; for if x^i_1 and x^i_2 are in $\overline{C}^{\alpha,i}_{n+i}$ then $|x^i_1-x^i_2| \subset |x^i_1| \cup |x^i_2|$, [6]. The factor group $C^{\alpha,i}_{n+i}/\overline{C}^{\alpha,i}_{n+i}$ is denoted by $\overline{\overline{C}}^{\alpha,i}_{n+i}$. Note that if $f(K) \subset A$ then $\overline{C}^{\alpha,i}_{n+i} = C^{\alpha,i}_{n+i}$.

Let $\{\overline{H}^{\alpha}_n\}$, $\{\overline{\overline{H}}^{\alpha}_n\}$ be the homology groups of the corresponding chain complexes $\{\overline{D}^{\alpha}_n, \overline{\Delta}^{\alpha}_n\}$, $\{\overline{\overline{D}}^{\alpha}_n, \overline{\overline{\Delta}}^{\alpha}_n\}$ where $\overline{D}^{\alpha}_n=\prod_i \overline{C}^{\alpha,i}_{n+i}$, $\overline{\overline{D}}^{\alpha}_n=\prod_i \overline{\overline{C}}^{\alpha}_{n+i}$ and $\overline{\Delta}^{\alpha}_n$, $\overline{\overline{\Delta}}^{\alpha}_n$ are the restrictions of Δ^{α}_n.

DEFINITION 3. The projective homology group of the space X (pair (X, A)) relative to A over the chain complex G is the direct limit $\underset{\overrightarrow{\alpha}}{Lim} \{\overline{H}^{\alpha}_n, \overline{\pi}_{\alpha\beta*}\}$ ($\underset{\overrightarrow{\alpha}}{Lim} \{\overline{\overline{H}}^{\alpha}_n, \overline{\overline{\pi}}_{\alpha\beta*}\}$), where $\alpha \in w(X)$; it is denoted by $\overline{H}_n(X;G)$ ($\overline{\overline{H}}_n(X, A;G)$).

It is easy to show that each of the pairs $\{\overline{H}_n(X;G), \bar{g}_*\}$ and $\{\overline{\overline{H}}_n(X, A; G), \bar{\bar{g}}_*\}$ is a covaricnt functor, where $\bar{g}_*$, $\bar{\bar{g}}_*$ are the induced, by the map $g: (X, A) \longrightarrow (Y, B)$ of the category Q, homomorphisms.

Along the following part denote by X the bouquet $X_1 \vee X_2$ of the two compact spaces X_1 and X_2, [7], and by A the subset X_2. Also, let $1: A \longrightarrow X$ be the inclusion map and $r: X \longrightarrow A$ the retraction, [5].

THEOREM 1. *The projective homology group of X relative to A over G is isomorphic to the projective homology group of the space X over G.*

PROOF. Define the homomorphism

$$T_n : H_n(A;G) \longrightarrow \overline{H}_n(X;G)$$

by : if $A_n \in H_n(A;G)$ and its representative is $h \in H^{\alpha}_n$, where $\alpha=(K;f) \in w(A)$, then $h \in H^{1(\alpha)}_n$ defines the element $T_n(A_n)$, where $1(\alpha)=(K;1f) \in w(X)$.

It is easy to prove that T_n is a monomorphism. In order to prove that T_n is an epimorphism, consider that $B_n \in \overline{H}_n(X;G)$ and $h \in \overline{H}^{\beta}_n$ such that h is a representative of B_n, where $\beta=(K_1\,;f_1)\in w(X)$. Assume that $Z_n\in\overline{D}_n^{\beta}$ belongs to the homology class h; i.e., $h=[Z_n]$. Let $K'_1=\bigcup_i K_i$, where $K_i=|Z^i|$ and Z^i is the i^{th} coordinate of Z_n. Thus we obtain such an element $\beta'=(K_1\,;f_1\pi)$ of the set $w(X)$ that $\beta'<\beta$ in virtue of the map π. Since $Z_n\in\overline{D}_n^{\beta'}$ it follows that $h'=[Z_n]\in\overline{H}_n^{\beta'}$. Moreover, h' is a representative of B_n.

On the other side, $h'\in H_n^{r(\beta')}$ and it defines an element A_n of the group $H_n(A\,;G)$, where $r(\beta')=(K'_1\,;rf_1\pi)$ belongs to $w(A)$. It remains to show that $T_n(A_n)=B_n$. Denote by $\hat{K}'_1$ the standard subdivision of $K'_1\times I$, where $I=[0,1]$. Define the map $g:K_1\longrightarrow X$ by: $g\theta_0=f_1$ and $g\theta_1=1r\,f_1\pi$, where $\theta_s(a)=(a,s)$ for every vertex a of K'_1 and $s=0,1$. It easy to show that g is a regular and $\breve{o}=(\hat{K}'_1\,;g)\in w(X)$. This follows, essentially, from the regularity of each Z^i relative to A (Definition 1). If the two pairs β', Ir (β') are identified with the corresponding pairs $(K'_1\times 0\,;g)$, $(K'_1\times 1\,;g)$ then $\beta'<\breve{o}$ and $1r(\beta')<\breve{o}$ in virtue of θ_0 and θ_1, respectively. Also we have $\theta_{0*}(h')=\theta_{1*}(h')$, i.e., $B_n=T_n(A_n)$.

THEOREM 2. *The projective homology group of the pair (X, A) relative to A over G is isomorphic to the projective homology group of the pair (X, A) over G.*

PROOF. Define the map

$$T^*_n : H_n(X,\ A\,;G)\longrightarrow\overline{\overline{H}}_n(X,\ A\,;G)$$

as follows. Let $A_n\in H_n(X,\ A\,;G)$ and $h\in H_n^{\alpha}$ such that $h\in A_n$, where $\alpha=(K,L\,;f)\in w(X,\ A)$. If 1_K is the identity map of the simplicial complex K onto itself then it induces a homomorphism $1_*:H_n^{\alpha}\longrightarrow\overline{\overline{H}}{}_n^{\alpha'}$, where $\alpha'=(K\,;f)\in w(X)$. The element $1_*(h)$ is a representative of $T_n(A_n)$.

Now the following sequence is exact, [8] :

$$N^{\alpha'}:\cdots\longrightarrow\overline{H}_n^{\alpha'}\longrightarrow H_n^{\alpha'}\longrightarrow\overline{\overline{H}}{}_n^{\alpha'}\longrightarrow\overline{H}_{n-1}^{\alpha'}\longrightarrow\cdots$$

Consider the following diagram:

$$\begin{array}{ccccccccc}
H_n(A\,;G) & \longrightarrow & H_n(X\,;G) & \longrightarrow & H_n(X,\ A\,;G) & \longrightarrow & H_{n-1}(A\,;G) & \longrightarrow & H_{n-1}(X\,;G)\\
\downarrow{\scriptstyle T_n} & & \| & & \downarrow{\scriptstyle T^*_n} & & \downarrow{\scriptstyle T_{n-1}} & & \|\\
\overline{H}_n(X\,;G) & \longrightarrow & H_n(X\,;G) & \longrightarrow & \overline{\overline{H}}_n(X,A\,;G) & \longrightarrow & \overline{H}_{n-1}(X\,;G) & \longrightarrow & H_{n-1}(X\,;G),
\end{array}$$

where the first row is the exact sequence of the projective homology groups over G (The fourth axiom of the triple $(\mathscr{H})$, the second row is the direct limit of the direct system $\{N^{\alpha'}\}$ over the set $w(X)$, [5], and T_n is the isomorphism of the theorem 1. It can be proved that this diagram is commutative and there by, the five lemma of homomorphisms, [5], suffice to prove the theorem.

The main result is the following theorem.

THEREM 3. *If U is an open subset of the space X (the bcuquet of X_1 and X_2) such that its closure $\overline{U}$ is contained in the interior of $A=X_2$,* i.e., *$\overline{U}\subset A^0$, the inclusion map $g:(X-U,\ A-U)\longrightarrow(X,\ A)$ induces the isomorphism $g_*:H_n(X-U,\ A-U\ ;G)\longrightarrow H_n(X,\ A\ ;G)$.*

PROOF. Let $X'=X-U$, $A'=A-U$ and O be the open covering of X consisting of A^0 and $X-\overline{U}$. We mention that $\overline{\overline{g}}_*\ T^*{}_n=T^*{}_n g_*$, wher $T^*{}_n$ is the isomorphism given in the theorem 2. Therefore, it is sufficient to prove that the map $\overline{\overline{g}}_*:\overline{\overline{H}}_n(X',\ A'\ ;G)\longrightarrow\overline{\overline{H}}_n(X,\ A\ ;G)$ is an isomorphism.

Firstly, we show that $\overline{\overline{g}}_*$ is an epimorphism. Let B_n be an element of the group $\overline{\overline{H}}_n(X,\ A\ ;G)$ with a representative $h\in\overline{\overline{H}}{}^{\alpha}_n$, where $\alpha=(K\ ;f)\in w(X)$. If the cycle $Z_n\in\overline{\overline{D}}{}^{\alpha}_n$ is belonging to the homology class h then it is not difficult to show that the i^{th} coordinate Z^i (for each i) of Z_n has such a representative $z^i\in C^{\alpha,i}_{n+i}$ that all f-images of the simplexes of z^i are contained in the covering O of X, i.e., they are contained in either A^0 or $X-U$. Consider that $z^i=z^i_1+z^i_2$, where z^i_1, z^i_2 are the restrictions of the chain z^i on those simplexes t of $|z^i|$ for which $f(t)\subset A^0$, $f(t)\subset(X-\overline{U})-(A^0\cap(X-\overline{U}))$, respectively. Assume that $K'=\{t\in K:f(t)\subset X-\overline{U}\}$ and $\alpha'=(K\ ;f|K')\in w(X')$. It is clear that $z^i_2\in C^{\alpha',i}_{n+i}$ and $\partial^i_{\alpha'}\ (z^i_2)\in\overline{C}{}^{\alpha',i}_{n+i-1}$. Thus we obtain such an element $\overline{h}=[\overline{Z}_n]\in\overline{\overline{H}}{}^{\alpha'}_n$ that z^i_2 is a representative of the i^{th} coordinate of Z_n. Moreover, $\overline{h}$ defines an elementn A_n of the group $\overline{\overline{H}}_n(X',\ A';\ G)$ for which $\overline{\overline{g}}_*(A_n)=B_n$.

Secondly, in order to prove that $\overline{\overline{g}}_*$ is a monomorphism, assume that $A_n\in\overline{\overline{H}}_n(X',\ A'\ ;G)$ and $h\in\overline{\overline{H}}{}^{\alpha}_n$ such that h is a representative of A_n, where $\alpha=(K\ ;f)\in w(X')$. Consider that $Z_n\in\overline{\overline{D}}{}^{\alpha}_n$ belongs to the homology class h. Let $z^i\in C^{\alpha,i}_{n+i}$ be a a representative of Z^i such that all gf-images of the simplexes of $|z^i|$ are contained in the covering O *of* X. Moreover, consider that $\overline{\overline{g}}_*(A_n)=0$.

This means that there exist a pair $\beta=(K_1; f_1)$ in the set $w(X)$ and an element X_{n+1} of the group $\overline{\overline{D}}^{\beta}_{n+1}$ such that $g(\alpha)<\beta$ in virtue of the inclusion $\pi: K\subset K_1$, and

$$\partial^i_\beta\, x^i+(-1)^{n+i}\, d^{i-1}_\beta\, x^{i-1}=\pi z^i+\bar{y}^i, \qquad \cdots (1)$$

(see [6]), where $x^i\in C^{\beta,i}_{n+i}$ is a representative of the i^{th} coordinate of X_{n+1} and $\bar{y}^i\in C^{\beta,i}_{n+i}$. Let $K'_1=\{t\in K_1: f_1(t)\subset X'\}$. It follows that the pair $\beta'=(K'_1; f_1|K'_1)$ belongs to $w(X')$ and $\alpha<\beta'$ in virtue of $\pi': K\subset K'_1$. Write

$$x^i=\sum_{k=1}^{3} x^i_k \text{ and } \bar{y}^i=\sum_{k=1}^{3}\bar{y}^i_k,$$

where x^i_2, x^i_3 ($\bar{y}^i_2$, $\bar{y}^i_3$) are the restrictions of the chain x^i ($\bar{y}^i$) on those simplexes of x^i ($\bar{y}^i$) for which $f_1(t)$ are contained in the covering O of X, $f_1(t)\subset A^0$, respectively; but x^i_1 ($\bar{y}^i_1$) denotes the restriction of x^i ($\bar{y}^i$) on the remain simplexes. The equality (1) can be rewritten as follows:

$$\partial^i_{\beta'}\, x^i_1+(-1)^{n+i}\, d^{i-1}_{\beta'}\, x^{i-1}_1=\pi' z^i+\bar{y}^i_1+c^i, \qquad \cdots (2)$$

where $c^i=\sum_{k=2}^{3}[\bar{y}^i_k-\partial^i_{\beta'}\, x^i_k+(-1)^{n+i+1}\, d^{i-1}_{\beta'}\, x^{i-1}_k]$.

It is clear that $\bar{y}^i_1+c^i\in\overline{C}^{\beta',i}_{n+i}$. If x^i_1 is considered as a representative of the i^{th} coordinate of an element $\overline{X}_{n+1}$ of the group $\overline{\overline{D}}^{\beta'}_{n+1}$ then the equality (2) implies that $\overline{\overline{\Delta}}^{\beta'}_{n+1}\,\overline{X}_{n+1}=\pi' Z_n$, i.e., $A_n=O$. This completes the proof of the theorem.

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REFERENCES

[1] Abd El-Sattar A. Dabbour, *Homotopy axiom for projective homology groups*; Bull. of Acad. of Sci. of Georgian SSR, 74 (1974), No. 3.

[2] Abd El-Sattar A. Dabbour, *On a generalization of Steenrod's homology groups*; Bull. of Acad. of Sci, of Georgian SSR, 77(1975), No. 2.

[3] Abd El-Sattar A. Dabbour, *Projective homology construction over a chain complex*; The 17 Annual Conf. on Stat. & comp. Sci. & Math., Cairo (1982).

[4] G. Chogoshvili, *Generalization of the product and limits and their applications in homology theory,* Uspehi Math. Nauk (YMH) USSR, 4(130), 1966.

[5] S. Eilenberg & N. E. Steenrod, *Foundation of algebraic topology,* Princeton University Prese, 1952.

[6] P. J. Hilton & S. Wylie, *Homology theory,* Cambridge, 1960.

[7] S. T. Hu, *Homotopy theory,* Academic Press, New York, 1959.

[8] S. Maclane, *Homology*, New York, 1967.

[9] N. Steenrod, *Regular cycles of compact metric spaces,* Ann. of Math., 2(41), 1940.