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AN EXAMPLE, WHICH SHOWS THAT THIERRIN'S CHARACTERIZATION FOR THE JACOBSON RADICAL OF A RING IS PROPERLY NEW

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In 1967, G. Thierrin [8] has introduced the notion of Neumannian right ideal R of a ring A, as follows:

A right ideal R is called in a ring A Neumannian, if for every element $a\in A$ there exists an element $b\in A$ such that

$$aba-a\in R$$

holds

In [8] is also verified by G. Thierrin that every m dular (see N. Divinsky [1], N. Jacobson [2] or F. Szàsz [6], [7]) (maximal) right ideal is Neumannian; furthermore the intersection of finite number of Neumannian right ideals is again Neumannian. Moreover, also by [8], if R is in A Neumannian, then the ideal quotient

$$R : a=\{x\ ;\ x\in A,\ \ ax\in R\}$$

is again Neumannian for arbitrary element $a\in A$.

One, among the important results of Thierrin [8], asserts that the Jacobson radical $J(A)$ of the ring A coincides with the intersection of all Neumannian (not necessarily maximal) right ideals.

THEOREM 1. (of F. Szàsz). *The Jacobson radical $J(A)$ of a ring A coincides with the intersection $T(A)$ of all Neumannian maximal right ideals R_α of A.*

RROOF (cf. [8]). By the first proposition of G. Thierrin [8] evidently $T(A)\leq J(A)$ holds, $J(A)$ being the intersection of all modular -maximal right ideals [1], [2], [6], [7], and since modularity of right ideals implies the Neumannian behaviour. If we assume that

$$T(A)\neq J(A)$$

is valid, then there exists an element $x\in J(A)$ such that $x\notin T(A)$. Then there exists also a Neumannian maximal right ideal R of A such that $x\notin R$. We now use the following right ideal:

$$\mathscr{K}=R\cap J(A).$$

Then there exists, by our assumption on R, an element $y\in A$ satisfying

$$xyx - x \in R,$$

whence $x \in J(A)$ implies also

$$xyx - x \in \mathscr{K}.$$

Furthermore, if $z = xy$, then

$$z^2 - z = (xyx - x) \cdot y \in \mathscr{K},$$

being $\mathscr{K}$ a right ideal in the ring A. Now $x \in J(A)$ implies $z \in J(A)$. Therefore there exists an element $w \in A$ such that:

$$z + w - z \cdot w = 0,$$

whence we obtain, by multiplication with $(-z)$, the equations:

$$z^2 \cdot w - z \cdot w - z^2 = 0,$$

and

$$(z^2 - z)w - z^2 = 0.$$

But $z^2 - z \in \mathscr{K}$ implies:

$$z^2 = (z^2 - z)w \in \mathscr{K}.$$

Thus

$$z = (z - z^2) + z^2 \in \mathscr{K},$$

consequently:

$$z \cdot x = xy \in \mathscr{K},$$

and

$$x = xyx - (xyx - x) \in \mathscr{K}$$

holds. Therefore:

$$x \in \mathscr{K} \subseteq R$$

is va id, which contradicts our assumption $x \notin R$. Consequently we obtain $T(A) = J(A)$, indeed, qu. e. d

THEOREM 2. *There exists a ring A containing a Neumannian, maximal, but non-modular right ideal R of A. Therefore the Thierrin-Szàsz characterization of the Jacobson radical $T(A) = J(A)$ of A is not only formally, but also properly new.*

PROOF. (cf. Szász [5], [6], [7]). The letter p let denote 0 or a prime number. Furthermore, let p_p be a prime field of characteristic p; $\mathscr{N}_\alpha$ be an arbitrary infnite cardinal number, Γ be a set of indexes of cardinality $\mathscr{N}_\alpha$, moreover $\delta_{\alpha\beta}$ be the Kronecker symbol. Let A be an algebra over P_p with the basis elements:

$$a_{\alpha},\ r_{\alpha\beta},\ s_{\alpha\beta\gamma} \qquad (\alpha,\ \beta,\ \gamma\in\Gamma)$$

and R be the subalgebra, generated over P_p by all $r_{\alpha\beta}$, $s_{\alpha\beta\gamma}$.

Put the multiplication table:

	a_ε	$r_{\varepsilon\eta}$	$s_{\varepsilon\eta\theta}$
a_α	a_ε	$\delta_{\alpha\varepsilon}\cdot a_\eta$	$\delta_{\alpha\varepsilon}\cdot a_\theta$
$r_{\alpha\beta}$	$s_{\alpha\beta\varepsilon}$	$\delta_{\beta\varepsilon}\cdot r_{\alpha\eta}$	$\delta_{\beta\varepsilon}\cdot s_{\alpha\eta\theta}$
$s_{\alpha\beta\gamma}$	$s_{\alpha\beta\varepsilon}$	$\delta_{\gamma\varepsilon}\cdot s_{\alpha\beta\eta}$	$\delta_{\gamma\varepsilon}\cdot s_{\alpha\beta\theta}$

Then A is a monomial algebra, in this basis, over P_p. Every element a of A has a form:

$$(*) \qquad a=\Sigma^*\ \pi_i\cdot a_{\alpha}+\Sigma^*\ \rho_{ij}\cdot r_{\alpha_i\beta_j}+\Sigma^*\ \sigma\quad\cdot s_{\alpha_i\beta_j\nu_k},$$

where π_i, ρ_{ij}, $\sigma_{ij\kappa}\in P_p$, and Σ^* is a finite sum.

We can explicitly verify that this table defines an associative multiplication.

Moreover, we obviously see that R is a right ideal of this ring A.

If $a\notin R$, $a\in A$, then there exists a coefficient $\pi_i\in P_p$ such that $\pi_i\neq 0$ in (*). Thus we have

$$a(\pi_i^{-1}\cdot\rho\cdot r_{\alpha_i\beta})=\rho\cdot a_{\beta}+r' \ \ (r'\in R)$$

for every $\beta\in\Gamma$ and $\rho\in P_p$, consequently:

$$aR+R=A.$$

This shows that R is a maximal right ideal in A, indeed.

Now we verify that R is not modular in the ring A, i.e. $(1-a)A\not\subseteq R$ holds for every $a\in A$. Namely, if $a\in R$, then

$$(1-a)a_{\alpha}=a_{\alpha}-a\cdot a_{\alpha}\notin R,$$

being $a_{\alpha}\notin R$. But, if $a\notin R$, then there exists in (*) a coefficient $\pi_i\neq 0$ such that

$$(1-a)\cdot(\pi_i^{-1}\cdot r_{\alpha_i\beta})=a_{\beta}+r''\notin R$$

Thus, by $(1-a)A\not\subseteq R$, the maximal right ideal R is not modular in A.

Now we show that R is a Neumannian right ideal in A. By $|\Gamma|=\mathscr{N}_{\alpha}$, there exists an index ω in (*), which is different from all occuring α_i, β_j, γ_κ. If all π_i is zero, then a in (*) belongs to R, and thus $a\in R$ trivially implies

$$a\cdot a\cdot a-a\in R.$$

But, if there exists for a in (*) a coefficient $\pi_i\neq 0$, then with the denoting:

$$(\Sigma^*\pi_i\cdot a_{\alpha_i}+\Sigma^*\ \rho_{ij}\cdot r_{\alpha_i\beta_j}+\Sigma^*\ \sigma_{ij\kappa}\cdot s_{\alpha_i\beta_j\gamma_\kappa})\cdot(\pi_i^{-1}\cdot s_{\alpha_i\mu\omega})=b$$

one calculates:

$$b=a_{\omega}+r'',\ \text{where } r''\in R\,;$$

whence we obtain, by (*) and by the choice of ω, the relations:

$$a(\pi^{-1}\ s_{\alpha_i\mu\omega})a-a=ba-a=(a_{\omega}\,rr'')a-a\in R.$$

Therefore R is a Neumannian maximal, but not modular right ideal of this ring A, qu. e. d.

REMARK 3. R is also *quasimodular* in A; that is $R:A\subseteq R$ holds, where

$$R:A=\{x\,;\,x\in A,\ Ax\subseteq R\}.$$

(See [5],]6], [7].)

REMARK 4. Over p_p holds:

$$\dim A=\dim R=\mathcal{N}_{\alpha}.$$

PROBLEM 5. Does there exist a Neumannian, maximal, but non-quasimodular right ideal of a ring?

PROBLEM 6. Does there exist a quasimodular, maximal, but non-Neumannian right ideal of a ring?

PROBLEM 7. (A. D. Sands, communication in a letter.). Is the Jacobson radical $J(A)$ the intersection of all simultenously left and right primitive ideals of the ring A?

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