Kyungpook Math. J.
Volume 23, Number, 2
December, 1983

NONHOMOGENEOUS BOUNDARY VALUE PROBLEMS FOR LINEAR MANIFOLDS II. ORDINARY DIFFERENTIAL SUBSPACES IN L_p-SPACES

By Sung J. Lee

1. Introduction

The nonhomogeneous boundary value problem for regular ordinary differential operators has been studied extensively. See, for example, the survey article [3]. The traditional way of studying this problem is by use of variation-of-constant formulas. In this paper, we will consider the corresponding problem for a multi-valued differential operator without using the variation-of-constant formula. The main idea is to treat problem as a special application of an abstract theorem in a Banach space developed in [5]. This theorem is similar to the Fredholm alternative, but does not require that a linear speac has a closed range. The underlying space in this paper is a L_p-type space ($1 \leq p < \infty$), and so only the Lagrange adjoint of a differential expression will play an important role. The same idea was used in [4] to study the same problem in the case when the underlying space is the Banach of continuous functions. The main results of this paper are Theorem 2.4 below for a regular differential subspace, and Theorem 3.1 for a singular differential operator. We now fix some notations. For a matrix D, its adjoint and transpose are denoted by D^* anp D^t, respectively. The adjoint of a linear manifold M is denoted by M^* (see [2] or [5]. For a notational convenience, the vector space of all $l \times k$ *row* matrices with complex entries is denoted by $\mathbb{C}^k$.

2. Regular Differential Subspaces

Let I be an interval. Let τ be the nth order expression

$$(\tau y)(x) := \sum_{k=0}^{n} p_k(x) y^{(k)}, \quad x \in I.$$

Here y is a $r \times l$ column vector, and p_k is *a* $r \times r$ matrix-valued function on I which is k times continuously differentiable (entrywise), and $p_n(x)$ is invertible for all $x \in I$. Let $p \in [1, \infty)$, $q \in [1, \infty)$, and let p' and q' be the conjugate of p and q, respectively. Let $L_p(I)$ be the Banach space of $r \times l$ column vector functions y defined on I such that

$$\|y\|_p := \left(\int_I (y^*y)^{p/2} dx\right)^{1/p} < \infty.$$

Let $T_0(\tau)$ and $T_1(\tau)$ be the graphs of the minimal and maximal closed differential operators generated by τ such that

$$T_0(\tau) \subset T_1(\tau) \subset \mathcal{L}_p(I) \oplus \mathcal{L}_q(I).$$

Throughout the rest of this section we will assume that I is a compact inteval $[a, b]$. Thus $T_0(\tau)$, $T_1(\tau)$ are regular differential operator. Let ω and ω^+ be d and d^+ finite dimensional vector spaces contained in $\mathcal{L}_p(I) \oplus \mathcal{L}_q(I)$ and $\mathcal{L}_{q'}(I) \oplus \mathcal{L}_{p'}(I)$, respectively, such that

$$\omega \cap T_1(\tau) = \{\{0, 0\}\}, \quad \omega^+ \cap T_0{}^*(\tau) = \{\{0, 0\}\}. \tag{2.1}$$

Let T be an arbitrary, but fixed closed vector space such that

$$^*(T_0{}^*(\tau) \dot{+} \omega^+) \subset T \subset T_1(\tau) \dot{+} \omega. \tag{2.2}$$

In this section, we will consider the problems:

(i) Estimate the dimensions of the null spaces of T and T^*.

(ii) For a given $g \in \mathcal{L}_q(I)$, consider the solvability problem of finding $f \in \mathcal{L}_p(I)$ such that $\{f, g\} \in T$.

If $d = d^+ = 0$, then the problems were solved in [1]. See the survey article [3] for a latest development. The problems when the underlying space is the Banach space of continuous functions, the corresponding problem was discussed in [4[. We will treat the problems as special cases of abstract problems in Banach space. Thus we will need following theorem which was proved in [5]. We will state it here.

THEOREM 2.1. *Let T_0, T_1 be the closed linear manifolds contained in the direct sum $X_1 \oplus X_2$ of Banach spaces X_1 and X_2 such that $T_0 \subset T_1$ and $N := \dim T_1/T_0 < \infty$. Let B be a continuous linear operator on T_1 onto $\mathbb{C}^N$ such that Null $B = T_0$, and B^+ be a w^*-continuous linear operator on $T_0{}^*$ onto $\mathbb{C}^N$ such that Null $B^+ = T_1{}^*$. Let C be the $N \times N$ invertible matrix such that*

Green's formula: $\bar{b}_2(a_2) - \bar{b}_1(a_1) = iB(a)C(B^+(b))^*$

for all $a = \{a_1, a_2\} \in T_1$, $b = \{b_2, b_1\} \in T_0{}^*$.

Finally let P be a $m \times N$ constant matrix of rank m, and define

$$T := \{a \in T_1 : P(B(a))^* = 0_{m \times l}\}.$$

Then we have the following:

[I] $m+dim\ Null\ T \geq dim\ Null\ T_1$,

$$N-m+dim\ Null\ T^* \geq dim\ Null\ T_0^*.$$

If we assume further that

$$dim\ Null T_0 = dim\ Null\ T_1^* = 0,$$

then

$$dim\ Null\ T_1 + dim\ Null\ T_0^* \leq N,$$

with the equality holding if and only if

$$dim\ Null\ T + dim\ Null\ T_0^* = dim\ Null\ T^* + N - m$$

and

$$dim\ Null\ T^* + dim\ Null\ T_1 = dim\ Null\ T + m.$$

[II] *Let* $g \in Range\ T_1$ *and* $\gamma \in \mathbb{C}^m$ *be given.*

(1) *If there exists* $s \in Dom\ T_1$ *such that*

$$(*)\ \{s, g\} \in T_1,\ \ P(B(\{s, g\}))^* = \gamma^t,$$

then

$$\overline{z(g)} = i\overline{\gamma}(PP^*)^{-1}PC(B^+(\{z, 0\}))^*$$

for all $z \in Null\ T_0^*$ *satisfying one of the following three equivalent conditions*:

(i) $z \in Null\ T^*$.

(ii) $B^+(\{z, 0\})$ *belongs to the vector subspace of* $\mathbb{C}^N$ *generated by the rows of* PC^{*-1}.

(iii) $C(B^+(\{z, 0\}))^* = P^*(PP^*)^{-1}PC(B^+(\{z, 0\}))^*$.

(2) *Assume further that*

$$dim\ Null\ T_0 = dim\ Null\ T_1^* = 0,$$

$$dim\ T_1/T_0 = dim\ Null\ T_1 + dim\ Null\ T_0^*.$$

Then the converse of (1) *is true.*

[III] *Assume that*

$$dim\ Null\ T_0 = dim\ Null\ T_1^* = 0,$$

$$dim\ T_1/T_0 = dim\ Null\ T_1 + dim\ Null\ T_0^*.$$

Then there exists a unique $s \in Dom\ T_1$ *satisfying* (*) *if and only if* $dim\ Null\ T=0$ *and* $m = dim\ Null\ T_1$.

REMARK. *In part* [III], *the "only if" was not stated in Theorem* 5 *of* [5]. *But this is obvious in view of part* [II] *as* $dim\ Null\ T=0$ *and* $m=dim\ Null\ T_1$

if and only if dim Null $T=0$, dim Null $T^*=0$.

We will change our problem in such a way that fits into the setting of the above theorem. Thus we must decide our maximal and minimal subspaces T_1, T_0, operators B, B^+, a corresponding $N\times N$ invertible matrix C. The corresponding Banach spaces in this case will be that

$$X_1=\mathcal{L}_p(I),\ X_2=\mathcal{L}_q(I).$$

To this end, we define

$$(2.3)\qquad T_1:=T_1(\tau)\dot{+}\omega,\ \ T_0:={}^*(T_0{}^*(\tau)\dot{+}\omega^+).$$

Then T_1 and $T_0{}^*$ are closed in $\mathcal{L}_p(I)\oplus\mathcal{L}_q(I)$ and $\mathcal{L}_{q'}(I)\oplus\mathcal{L}_{p'}(I)$, respectively. Because of (2.1) we can rewrite ω, ω^+ as

$$(2.4)\quad \omega=\{\{W_1\,\alpha^t,\ W_2\,\alpha^t\}\,|\,\alpha\in\mathbb{C}^d\},\ \ \omega^+=\{\{W_1{}^+\,\alpha^t,\ W_2{}^+\alpha^t\}\,|\,\alpha\in\mathbb{C}^{d+}\},$$

where W_1 and W_2 are $r\times d$ matrices whose columns are in $\mathcal{L}_p(I)$ and $\mathcal{L}_q(I)$, respectively, such that whenever $\{W_1\,\alpha^t,\ W_2\,\alpha^t\}\in T_1\ (\tau)$ for some $\alpha\in\mathbb{C}^d$, then $\alpha=0_{1\times d}$. Similarly, $W_1{}^+$ and $W_2{}^+$ are $r\times d^+$ matrices whose columns are in $\mathcal{L}_{q'}(I)$ and $\mathcal{L}_{p'}(I)$, respectively, such that whenever $\{W_1{}^+\,\alpha^t,\ W_2{}^+\,\alpha^t\}\in T_0{}^*\,(\tau)$ for some $\alpha\in\mathbb{C}^{d+}$, then $\alpha=0_{1\times d}$.
Since it is well known that
dim $(T_1(\tau)/T_0(\tau))=2$ rn, it follows from

$$*(T_0{}^*\,(\tau)\dot{+}\omega^+)\subset T_1(\tau)\dot{+}\omega$$

that

$$(2.5)\qquad N:=\dim\ (T_1/T_0)$$
$$=d+d^++2rn.$$

We will now proceed to construct boundary operators B and B^+. First we have

PROPOSITION 2.2. (i) *There exists a $r\times d$ matrix $'W$ such that the columns of $'W$ are in* Dom $T_1{}^*(\tau)$ *and*

$$\int_I(W_1{}^t\tau^+\overline{('W)}-W_2{}^t\overline{'W})dx=I_{d\times d}.$$

Moreover, any $w=\{w_1,\ w_2\}\in\omega$ can be written uniquely as

$$w=\{W_1\,\alpha^t,\ W_2\,\alpha^t\}\ \textit{where}$$

$$\alpha=\int_I (W_1{}^t\overline{\tau^+('W)}-W_2{}^t\overline{'W})\,dx.$$

(ii) *There exists a $r\times d^+$ matrix $'W^+$ such that the columns of $'W^+$ are in Dom $T_0(\tau)$ and*

$$\int_I((W_1{}^+)^t\overline{\tau('W^+)}-(W_2{}^+)^t\overline{('W^+)})dx=I_{d^+\times d^+}.$$

Moreover, any $w^+=\{w_1{}^+,\ w_2{}^+\}\in\omega^+$ can be written uniquely as

$$w^+=\{W_1{}^+\ \alpha^t,\ W_2{}^+\ \alpha^t\}\ \textit{where}$$

$$\alpha=\int_I((w_1{}^+)^t\overline{\tau^+('W^+)}-(w_2{}^+)^t\overline{('W^+))}\ dx.$$

PROOF. (i) The map

$$\{z,\ \tau^+z\}\longrightarrow\int_I((\tau^+z)^t\overline{W}_1-z^t\overline{W}_2)\ dx$$

defines a map on $T_1{}^*(\tau)$ onto $\mathbb{C}^d$. For, suppose that for some $\beta\in\mathbb{C}^d$,

$$\int_I((\tau^+z)^t\overline{W}_1-z^t\overline{W}_2)\ dx\ \beta^*=0$$

for all $z\in$ Dom $T_1{}^*(\tau)$. Then $\beta=0_{1\times d}$ as

$$T_1(\tau)={}^*(T_1{}^*(\tau)),\ \ \{W_1\beta^t,\ W_2\beta^t\}\in T_1(\tau).$$

This shows that the map is surjection. This shows the existence of such a $'W$. Take any $w=\{w_1,\ w_2\}\in\omega$. Then, since the columns of $\{W_1,\ W_2\}$ form a basis for ω, there exists a unique $\alpha\in\mathbb{C}^d$ such that

$$w=\{W_1\alpha^t,\ W_2\alpha^t\}.\ \text{But, then}$$

$$\int_I(w^t\overline{\tau^+('W)}-w_2{}^t\overline{'W_1})\ dx$$

$$=\int_I(\alpha W_1{}^t\ \overline{\tau^+('W)}-\alpha\,W_2{}^t\overline{'W_1})\ dx=\alpha.$$

This proves (i). The proof for (ii) is similar. This completes the proof.

For $y\in$ Dom $T_1(\tau)$, or in Dom $T_0{}^*(\tau)$, let $\tilde{y}(x)$ be the $rn\times 1$ column matrix defined by

$$(\tilde{y}(x))^t=((y(x))^t,\ (y'(x))^t,\ \cdots,\ (y^{(n-1)}(x))^t).$$

Then there exists a $rn\times rn$ invertible matrix $C_1(x)$ depending only on the values of the coefficients of τ and its Lagrange adjoint, τ^+, of τ such that

$$(2.6)\qquad \int_I (z^*(x)(\tau y)(x)-(\tau^+ z)^*(x)\,y(x))\,dx$$

$$=i\begin{pmatrix} y(a) \\ y(b) \end{pmatrix}^t \left(\begin{array}{c:c} iC_1(a) & 0_{rn\times rn} \\ \hdashline 0_{rn\times rn} & -iC_1(b) \end{array}\right) \overline{\begin{pmatrix} z(a) \\ z(b) \end{pmatrix}}$$

Here $'W$ and $'W^+$ are the same as in Proposition 2.2.

Let C be the $N\times N$ invertible block matrix defined by

$$(2.9)\qquad C:=\left(\begin{array}{c:c} iC_1(a) & 0_{rn\times rn} \\ \hdashline 0_{rn\times rn} & -iC_1(b) \end{array}\right)\dot{+}\left(-iI_{d^*\times d^*}\right)\dot{+}\left(iI_{d\times d}\right)$$

PROPOSITION 2.3. *Let B, B^+, C be as (2.7)−(2.9). Then B and B^+ defines continuous linear operators on T_1 and T_0^* onto $\mathbb{C}^N$, respectively, such that*

$$\text{Null } B=T_0^*,\ \text{Null } B^+=T_1^*.$$

Moreover, we have the Green's formula:

$$\int_I [(\tau y+w_2)^t\,\overline{(z+w_1^{\,+})}-(y+w_1)^t\,\overline{(\tau^+ z+w_2^{\,+})}]\,dx$$

$$=iB(\{y+w_1,\ \tau y+w_2\}\ C(B^+(\{z+w_1^{\,+},\ \tau^+ z+w_2^{\,+}\}))^*$$

for all $y\in$ Dom $T_1(\tau)$, $z\in$ Dom $T_0^*(\tau)$, $\{w_1, w_2\}\in\omega$ and $\{w_1^{\,+}, w_2^{\,+}\}\in\omega$.

PROOF. It is clear that B is continuous on T_1 into $\mathbb{C}^N$, and B^+ is continuous on T_0^* into $\mathbb{C}^N$. It is also clear that Null $B=T_0$, Null $B^+=T_1^*$. We will show that Range $B=\mathbb{C}^N$. To this end, for all $y\in$ Dom $T_1(\tau)$, $z\in$ Dom $T_0^*(\tau)$. For this, see p 73 of [2].

We now define the boundary operators B and B^+ on T_1 and T_0^* as follows: Let B be the linear operator on T_1 into $\mathbb{C}^N$ and B^+ the one on T_0^* into $\mathbb{C}^N$ defined by

$$(2.7)\quad B(\{y,\ \tau y\}+\{w_1,\ w_2\})$$

$$=[(\tilde{y}(a))^t,\ (\tilde{y}(b))^t,\ \int_I ((\tau y)^t\overline{W_1}-y^t\overline{W_2^{\,+}})\,dx,$$

$$\int_I (W_1^{\,t}\overline{\tau^+('W)}-w_2^{\,t}\overline{'W})\,dx]$$

for $\{y,\ \tau y\} \in T_1(\tau),\ \{w_1,\ w_2\} \in \omega$:

(2.8) $$B^+ (\{y,\ \tau^+ y\} + \{w_1^+,\ w_2\}) = [(\tilde{y}(a))^t,\ (\tilde{y}(b))^t,\ \int_I ((w_1^+)^t \overline{\tau('w^+)} - (w_2^+)^t \overline{'w^+})\, dx,\ \int_I ((y+w_1^+)^t \overline{w_2} - (\tau^+ y + W_2^+)^t \overline{w_1})\, dx]$$

for $\{y,\ \tau^+ y\} \in T_0^*(\tau),\ \{w_1^+,\ w_2^+\} \in \omega^+$.

it is sufficient to show that $(\text{Range } B)^{\perp} = \{0\}$. Let $\alpha,\ \beta,\ \gamma$ be the $1\times 2rn,\ 1\times d^+$, $1\times d$ constant matrices

$$B(\{y+w_1,\ \tau y+w_2\})\ (\alpha,\ \beta,\ \gamma)^* = 0$$

for all $y \in \text{Dom } T_1(\tau),\ \{w_1,\ w_2\} \in \omega$.

Then

$$((\tilde{y}(a))^t,\ (\tilde{y}(b))^t)\ \alpha^* = 0,$$

$$\int_I [(\tau y)^t \overline{w_1^+} - y^t \overline{w_2^+}]\ dx\beta^* = 0,$$

$$\int_I [w_1^t \overline{\tau^+('w)} - w_2^t \overline{'w}\ dx]\ \gamma^* = 0$$

for all $y \in \text{Dom } T_1(\tau),\ \{w_1,\ w_2\} \in \omega$.

It is well known that

$$\{[(\tilde{y}(a))^t,\ (\tilde{y}(b))^t] : y \in \text{Dom } T_1(\tau)\} = \mathbb{C}^{2rn}.$$

It follows that $\alpha = 0_{1\times 2rn}$

The second equation in the above implies that

$$\{w_1^+ \beta^t,\ w_2^+ \beta^t\} \in T_0^*(\tau),$$

and hence $\beta = 0_{1\times d^+}$. The third equation holds, in particular, for $w_1 = w_1^+ \gamma^t$, $w_2 = W_2^+ \gamma^t$, and hence $\gamma\ I_{d\times d}\ \gamma^* = 0$. Thus $\gamma = 0_{1\times d}$. This shows that Range $B = \mathbb{C}^N$. Similarly, we can show that Range $B^+ = \mathbb{C}^N$. To prove the Green's formula, let $y \in \text{Dom } T_1(\tau),\ z \in \text{Dom } T_0^*(\tau),\ w \in \omega,\ w^+ \in \omega^+$. By Proposition 2.2, we can write

$$w = \{w_1 \alpha^t,\ w_2 \alpha^t\},\ w^+ = \{w_1^+ \beta^t,\ w_2^+ \beta^t\},$$

where

$$\alpha = \int_I [w_1^t \overline{\tau^+ ('w)} - w_2^t \overline{'w}]\ dx,$$

$$\beta=\int_I [(w_1^+)^t \tau('W^+) - (w_2^+)('W^+)]\ dx.$$

Using the expressions for w and w^+ as the above and the relation in (2.6), we can easily verify the Green's formula.

We will now rewrite T in (2.2) as in Theorem 2.1. Since T is closed, there exists a $m\times N$ constant matrix P of rank m such that

(2.10) $$T=\{\{y+w_1,\ \tau y+w_2\}\,|\,y\in \text{Dom } T_1(\tau),\ \{w_1,\ w_2\}\in\omega,\ P(B(\{y+w_1,\ \tau y+w_2\}))^*=0_{m\times 1}\}.$$

The condition for (2.10) is rewritten equivalently as follows:

(2.11) $$0_{m\times N}=P_1\overline{\tilde{y}(a)}+P_2\overline{\tilde{y}(b)} + R\int_I (\tau y)^t\overline{w_1^+} - y^t\overline{w_2^+}]^*\ dx + S\int_I [w_1{}^t\overline{{}_\tau^+('w)} - w_2{}^t\overline{'w}]^*\ dx,$$

where P_1, P_2, R, S are the $m\times rn$, $m\times rn$, $m\times d^+$, $m\times d$ constant matrices such that $P=[P_1,\ P_2,\ R,\ S]$ and P is of rank m.

The last integral can be rewritten as α if $w_1 = w_1\alpha^t$, $w_2 = w_2\alpha^t$ for some $\alpha\in\mathbb{C}^d$.

Now, with T_1, T_0, B, B^+, C defined in (2.3), (2.7)–(2.9), we are now in a position to apply Theorem 2.1 to the present setting.

THEOREM 2.4. *Let T be as* (2.10). *Then we have the following* [I]–[II].

[I] dim Null $T\le d+d^++2rn-m$,
dim Null $T^*\le m$,
dim Null $T^*=$dim Null $T+m-d-rn$.

[II] *Let $g\in\mathcal{L}_q(I)$, $\gamma\in\mathbb{C}^m$ be given. Define*

$$\Gamma(z,\ \beta) := P_1C_1\ (a)\ \overline{\tilde{z}(a)} - P_2C_1\ (b)\ \overline{\tilde{z}(b)} - R\beta^* + S\int_I w_2{}^t\overline{(z+w_1{}^+\beta^t)}\ dx$$

for $z\in$ Dom $T_0^*(\tau)$, $\beta\in\mathbb{C}^{d^+}$. Then in order that there exists $y\in$ Dom $T_1(\tau)$ and $\alpha\in\mathbb{C}^d$ such that

(1) $(\tau y)(x)+w_2(x)\alpha^t=g(x)$, *a. a.* $x\in I$,

(2) $$P_1\overline{\overline{y}(a)}+P_2\overline{\overline{y}(b)}+S\alpha^*$$
$$+R\int_I [(w_1^{\ +})^t\overline{\tau y}-(w_2^{\ +})^t\overline{y}]\ dx$$
$$=\gamma^t,$$

it is necessary and sufficient that

(3) $$\int_I g^t(x)\ \overline{(z+w_1^{\ +}\beta^t)}\ (x)\ dx=-\gamma(PP^*)^{-1}\Gamma(z.\ \beta)$$

for all $z\in \mathrm{Dom}\ T_0^*\ (\tau)$ *and* $\beta\in \mathbb{C}^{d^*}$

which satisfy

$$(\tau^+z)(x)+w_2^{\ +}(x)\,\beta^t=0,\ \ a.\ a.\ x\in I$$

and one of the following three equivalent conditions:

(i) $z+w_1^{\ +}\beta^t\in \mathrm{Null}\ T^*.$

(ii) *There exists* $\delta\in\mathbb{C}^m$ *such that*

$$\bar{z}^t(a)=\delta P_1(C_1^*(a)^{-1},\ \ \bar{z}^t(b)=\delta P_2(C_1^*(b)^{-1},$$
$$\beta=-\delta R,\ \int_I(z+W_1^{\ +}\beta^t)\ \overline{W_2}dx=\delta S.$$

(iii) $C_1\ (a)\ \overline{\bar{z}(a)}=P_1^*\,(PP^*)^{-1}\Gamma(z,\ \beta),$

$$-C_1\,(b)\ \overline{\bar{z}(b)}=P_2^*(PP^*)^{-1}\Gamma(z,\ \beta),$$
$$-\beta^*=R^*(PP^*)^{-1}\Gamma(z,\ \beta),$$
$$\int_I w_2^{\ t}\overline{(z+W_1^{\ +}\beta^t)}\ dx=S^*(PP^*)^{-1}\Gamma(z,\ \beta).$$

〔III〕. *If Null* $T=\{0\}$, *and* $m=d+rn$, *then there exists a unique pair* $\{y,\ \alpha\}$ *with* $y\in \mathrm{Dom}\ T_1(\tau)$, $\alpha\in\mathbb{C}^d$ *satisfying* (1) *and* (2), *and conversely.*

PROOF. It is clear that

$$\dim\ \mathrm{Null}\ T_0=\dim\ \mathrm{Null}\ T_1^*=0.$$

Since

$$\dim\ T_1(\tau)\cap\omega=\dim\ T_0^*(\tau)\cap\omega^+=0,$$

we see that

$$\dim\ \mathrm{Null}\ T_1=d+rn,\ \ \dim\ \mathrm{Null}\ T_0^*=d^++rn.$$

Thus

$$N = \dim \text{ Null } T_1 + \dim \text{ Null } T_0^* = 2rn + d + d^+.$$

Therefore the results of [I] follows from [I] of Theorem 2.1. To prove [II], we let $X_1 = \mathcal{L}_p(I)$, $X_2 = \mathcal{L}_q(I)$, and C be as (2.9). Thus [II] of Theorem 2.1 will be applicable here. In this situation,

$$\bar{b}_2(a_2) - b_1(a_1) = \int_I a_2^{\ t}(x)\bar{b}_2(x)dx - \int_I a_1^{\ t}(x)\bar{b}_1(x)dx$$

for all $\{a_1, a_2\} \in \mathcal{L}_p(I) \oplus \mathcal{L}_q(I)$, $\{b_2, b_1\} \in \mathcal{L}_{q'}(I) \oplus \mathcal{L}_{p'}(I)$.

Since Range $T_1(\tau) = \mathcal{L}_q(I)$, Range $T_1 = \mathcal{L}_q(I)$.

Thus $g \in$ *Range* T_1 if and only if $g \in \mathcal{L}_q(I)$.

Because of (2.10), (2.11) and since $T_1 = T_1(\tau) \dot{+} \omega$, the statement in Theorem 2.1 that there exists $s \in$ *Dom* T_1 such that

$$\{s, g\} \in T_1, \ \ P(B(\{s, g\}))^* = \gamma^t$$

is equivalent to the existence of $y \in$ *Dom* $T_1(\tau)$, $\alpha \in \mathbb{C}$ satisfying (1) and (2). In fact, $s = y + w_1\alpha^t$.

The statement in Theorem 2.1 that $\{u, 0\} \in T_1$ is equivalent to the statement that $u = z + w_1^{\ +}\beta^t$ for some $z \in$ *Dom* $T_1(\tau)$ and $\beta \in \mathbb{C}^{d^+}$ such that $\tau^+ z + w_2^{\ +} \beta^t = 0$ in $\mathcal{L}_{p'}(I)$. The term $iPC(B^+(\{u, 0\}))^*$ in Theorem 2.1 coincides with $\Gamma(z, \beta)$ where in this case

$$u = z + W_1^{\ +}\beta^t \ \ (z \in Do\,:\, T_1(\tau), \ \ \beta \in \mathbb{C}^{d^+}).$$

It follows that the statement

$$\bar{u}(g) = i\gamma(PP^*)^{-1}PC(B^+\{u, 0\})^*$$

for all $u \in$ *Dom* T_0^* is equivalent to the statement in (3) for all $z \in$ *Dom* $T_0^*(\tau)$ and $\beta \in \mathbb{C}^{d^+}$ satisfying $\tau^+ z + w_2^{\ +} \beta^t = 0$. In fact, $u = z + w_1^{\ +}\beta^t$. That $u \in$ Null T^* in Theorem 2.1 is equivalent to $u = z + W_1^{\ +}\beta^t \in$ Null T^*. The statement (ii) in Theorem 2.1 is equivalent to (ii) of the above theorem. The statement (iii) of Theorem 2.1 is equivalent to the statement in (iii) of the above theorem. Therefore by Theorem 2.1 Part [II] is valid. Assume now that dim Null $T = 0$ and $m = d + rn$. Then by Part [I], dim Null $T^* = 0$. If $\beta \in \mathbb{C}^{d^+}$, and $z \in$ *Dom* $T_0^*(\tau)$ such that $\tau^+ z + w_2^{\ +}\beta^t = 0$ and $z + w_1^{\ +}\beta^t \in$ Null T^*, Then $\{w_1^{\ +}\beta^t, \ w_2^{\ +}\beta^t\} = -\{z, \tau^+ z\}$, and so $\beta = 0_{1\times d^+}$ and hence $z = 0$. It then follows that the left and right hand side of (3) are 0. Thus by Part [II], there exists a unique pair $\{y, \alpha\} \in Dom\ T_1(\tau)\oplus \mathbb{C}^d$ satisfying (1) and (2), and this must be unique as dim Null $T=0$. This completes the proof.

COROLLARY 2.5 (Coddington and Levinson [1]).
Let

$$T := \{\{y, \tau y\} : y \in Dom\ T_1(\tau),$$
$$P_1\,\overline{\hat{y}(a)}+P_2\,\bar{\hat{y}}(b)=0_{m\times 1}\},$$

where P_1 and P_2 are $m\times rn$ constant matrices such that the $m\times 2rn$ composition matrix $[P_1, P_2]$ is of rank m. Then

[I] dim Null $T^*=$dim Null $T+m-rn$.

[II] *Let $g\in \mathcal{L}_p(I)$ and $\gamma\in \mathbb{C}^m$ be given. Then in order that there exists y in $Dom\ T_1(\tau)$ such that*

$$(*)\ \tau y=g \text{ in } \mathcal{L}_q(I),\ P_1\,\overline{\hat{y}(a)}+P_2\overline{\hat{y}}(b)=\gamma^t,$$

it is necessary and sufficient that

$$\int_I g^*(x)\ \overline{z(x)}dx=\gamma(P_1P_1^*+P_2P_2^*)^{-1}\ [P_2C_1(b)\ \overline{\hat{z}(b)}$$
$$-P_1C_1(a)\,\bar{\hat{z}}(a)]$$

for all $z\in$ Null T^*.

[III] *If* dim Null $T=0$ *and $m=rn$, then there exists a unique element $y\in Dom\ T_1(\tau)$ satisfying* (*), *and conversely.*

PROOF. In Theorem 2.4, take $\omega=\{\{0, 0\}\}$, $\omega^+=\{\{0, 0\}\}$.

REMARK. Coddington and Levinson [1] considered the case when $r=1$. Part [I], Part [II] and Part [III] are Theorem 3.4, Theorem 4.1 and Corollary (p. 295) of [1]. The term $\gamma\cdot U_1^{+}\Psi$ in p. 294 of [1] coincides with the right hand side of the above integral. This is much more clearer than that of [1].

3. Singular nonhomogeneous boundary value problem

In this section we will consider the problems (i) and (ii) mentioned in §2 in the case when I is a semi-infinite interval. Thus in this section $I:=[a, \infty)$ ($|a|<\infty$). Let $T_0(\tau)$, $T_1(\tau)$ will be as in §2. We will consider the case when the boundary conditions at $+\infty$ disappear. Thus we assume that

(3.1) $$\dim T_1(\tau)/T_0(\tau)=rn.$$

Since $x=a$ is a regular point for $T_1(\tau)$, this assumption is equivalent to

(3.2) $$\lim_{x\to\infty} \tilde{y}^t(x)C_1(x)(\tilde{z}^t(x))^*=0$$

for all $y\in Dom\ T_1(\tau),\ z\in Dom\ T_0^*(\tau)$.

The proof for this can be carried out as in the proof of Theorem 3.1 of [6].

THEOREM 3.1. *Assume* (3.1). *Let*

$$T:=\{\{y,\tau y\}\,|\,y\in Dom\ T_1(\tau),\ P\ \overline{\tilde{y}(a)}=0_{m\times 1}\},$$

where P is a $m\times rn$ constant matrix of rank m. Then

[I] dim Null $T\leq rn-m$, dim Null $T^*\leq m$.

Moreover,

$$\text{dim Null } T_1(\tau)+\text{dim Null } T_0^*(\tau)=rn$$

if and only if

$$\begin{aligned}\text{dim Null } T-\text{dim Null } T^* &= rn-m-\text{dim Null } T_0^*(\tau)\\ &= \text{dim Null } T_1(\tau)-m.\end{aligned}$$

[II] *Assume that*

$$\text{dim Null } T_1(\tau)+\text{dim Null } T_0^*(\tau)=rn.$$

Let $g\in$ Range $T_1(\tau)$ and $\gamma\in\mathbb{C}^m$ be given. Then in order that there exists $y\in$ Dom $T_1(\tau)$ such that

$$(*)\quad \tau y=g,\ P\ \overline{\tilde{y}(a)}=\gamma^t$$

it is necessary and sufficient that

$$\int_a^\infty z^*(x)g(x)\,dx=-\bar{\gamma}(PP^*)^{-1}C_1(a)\ \overline{\tilde{z}(a)}$$

for all $z\in$ Null $T_0^*(\tau)$ satisfying one of the following three equivalent conditions.

(i) $z\in$ Null T^*.

(ii) *There exists $\delta\in\mathbb{C}^m$ such that*

$$(\tilde{z}(a))^t=\delta\, P(C_1^*(a))^{-1}.$$

(iii) $C_1(a)\overline{\tilde{z}(a)}=P^*(PP^*)^{-1}PC_1(a)\overline{\tilde{z}(a)}$.

[III] *Assume that*

$$rn=\text{dim Null } T_1(\tau)+\text{dim Null } T_0^*(\tau).$$

Then there exists a unique $s\in Dom\ T_1(\tau)$ satisfying (*) *if and only if* dim Null

$T=0$ and $m=\dim \text{Null } T_1(\tau)$.

PROOF. In Theorem 2.1, let $X_1=\mathcal{L}_p(I)$, $X_2=\mathcal{L}_q(I)$, $T_1=T_1(\tau)$, $T_0=T_0(\tau)$, and C_1 be as in §2. Finally let

$$B(\{y, \tau y\}) := (\tilde{y}(a))^t, \quad y \in Dom\ T_1(\tau),$$

$$B^+(\{y, \tau^+ y\}) : (\tilde{y}(a))^t, \quad y \in Dom\ T_0^*(\tau).$$

The Green's formula in Theorem 2.1 now takes the form:

$$\int_a^\infty [(\tau y)^t \bar{z} - y^t \overline{(\tau^+ z)}]\ dx$$

$$= iB(\{y, \tau y\}) C_1(a) (B^+(\{z, \tau^+ z\}))^*$$

for all $y \in Dom\ T_1(\tau)$, $z \in Dom\ T_0^*(\tau)$. The operator B defines a continuous linear operator on $T_1(\tau)$ onto $\mathbb{C}^{rn}$ with Null $B=T_0$, and B^+ a continuous linear operator on $T_0^*(\tau)$ onto $\mathbb{C}^{rn}$ with Null $T_1^*(\tau)$.

Now, since

dim Null $T_0(\tau)=$dim Null $T_1^*(\tau)=0$, the result follows from Theorem 2.1.

REMARK. In Theorem 2.4 and 3.1 we discussed a solvability problem of nonhomogeous boundary value problem. The actual construction of the solutions can be worked out using the results of [7].

University of south Florida
Tampa, FL 33620

REFERENCES

[1] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, Mc-Graw-Hill (1955), New York.

[2] E. A. Coddington and A. Dijksma, *Adjoint subspaces in Banach spaces, with application to ordinary differential subspaces*, Anal. Mat. Pura Appl., (4) 118 (1978), 1—118.

[3] A. M. Krall, *The development of general differential and general differential-boundary systems*, Rocky Mountain J. Math., (5) 4 (1975), 493—542.

[4] S. J. Lee, *Existence of the solutions of bounded variations to linear integro-differential equations*, J. Diff. Equations, (42) 1(1981), 129—153.

[5] S. J. Lee, *Nonhomogeneous boundary value problemes for linear manifolds I*, J. Math. Anal. Appl. (in press).

[6] S. J. Lee, *On boundary conditions for linear differential operators*, J. London Math. Soc., (2) 12 (1976), 447—454.

[7] S. J. Lee and M. Z. Nashed, *Generalized inverses for linear manifolds and applications to boundary problems in Banach spaces*, Math. Report. Acad. Sciences of Canada (in press)

[8] S. Schwabick, M. Tvrdy and O. Vejvada, *Differential and Integral Equations*, D. Reidel, Boston (1979).