Kyungpook Math. J.
Volume 23, Number 1
June, 1983

PARSEVAL'S IDENTITY ON BANACH SPACES

By M. A. Kandil

1. Introduction

Parseval's identity plays a central role in the theory of Fourier's coefficients in Hilbert spaces. An extension of the identity to the space of vector valued functions with values in a Banach space seems in order.

In this article, we carry out the treatment for two cases. The first, are square B integrable vector valued functions, with values in a Hilbert space. The second, which is more interesting, is a square B integrable, vector valued function with values in a Banach space. Also, we show that Parseval's identity may not be considered as a test for Hilbertization of a Banach space of values for square B integrable vector valued functions.

2. Parseval's identity for $B_2(H, K_2)$

For simplicity, we here consider vector valued functions of two variables on the Euclidean space R_2. The result can be generalized directly to the n-dimensional Euclidean space R_n. So, let

(i) K_2 be a two dimensional rectangle,

$$K_2=\{0\leq x_1\leq 2\pi,\ 0\leq x_2\leq 2\pi\}$$

(ii) $B_2(X, K_2)$ be the space of square B integrable vector valued functions on K_2 to a Banach space X, with the norm $\|\ \|$.

Hence, if

$$f(x_1, x_2)\in B_2(X, K_2).$$

Then
$$\int_0^{2\pi}\int_0^{2\pi} \|f(x_1, x_2)\|^2\, dx_1\, dx_2<\infty$$

(iii) Fourier's coefficient for vector valued functions in $B_2(X, K_2)$ can be defined as

$$c_{k_1,k_2}=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(x_1, x_2)e^{-i(k_1x_1+k_2x_2)}\, dx_1\, dx_2$$

where $k_1, k_2=0, \pm 1, \pm 2, \ldots,$

(iv) $B_2(H, K_2)$ be a special case of $B_2(X, K_2)$ with the values in a Hilbert space H.

On analogy of the inner product of Hilbert spaces, let us introduce for any two functions $f(x_1, x_2)$, $g(x_1, x_2)$ in $B_2(H, K_2)$ the following inner product,

$$<f, g>=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}(f(x_1, x_2),\ g(x_1, x_2))\, dx_1\, dx_2.$$

So, $B_2(H, K_2)$ becomes a Hilbert space of vector valued functions, where $(f(x_1, x_2), g(x_1, x_2))$ is the usual inner product on the Hilbert space H.

LEMMA. *Let* $\{e_1, e_2, \ldots, e_p, \ldots\}$ *be a complete orthonormal system in* H. *Then* $\{e_p e^{i(k_1x_1+k_2x_2)}\}$, $p=1, 2, 3, \ldots$; $k_1, k_2=0, \pm 1, \pm 2, \ldots,$ *is a complete orthonormal system in* $B_2(H, K_2)$.

PROOF. It is required to prove that, for every combination (p, k_1, and k_2) if $<f(x_1, x_2),\ e_p e^{i(k_1x_2+k_2x_2)}>=0$, then $f(x_1, x_2)=\theta$ almost everywhere on K_2, where θ is the zero vector for the space H. So,

$$<f(x_1, x_2),\ e_p e^{i(k_1x_1+k_2x_2)}>=$$

$$=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}(f(x_1, x_2),\ e_p e^{i(k_1x_1+k_2x_2)})\, dx_1\, dx_2$$

$$=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}(f(x_1, x_2),\ e_p)e^{-i(k_1x_1+k_2x_2)})\, dx_1\, dx_2 \tag{1}$$

The numerical function $(f(x_1, x_2), e_p)$ is defined on K_2 to R_1 (R_1 is the real line) and summable. Hence if (1) equal zero we have $(f(x_1, x_2), e_p)=0$ almost everywhere on K_2, consequently the lemma.

THEOREM. *If* $f(x_1, x_2)\in B_2(H, K_2)$ *and if* c_{k_1,k_2} *are its Fourier coefficients then,*

$$<f(x_1, x_2),\ f(x_1, x_2)>=\sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty}(c_{k_1k_2},\ c_{k_1k_2})$$

PROOF. Applying the previous lemma, we have

$$<f(x_1, x_2), f(x_1, x_2)>$$
$$= \sum_{p=1}^{+\infty} \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} |<f(x_1, x_2), e_p e^{i(k_1x_1+k_2x_2)}>|^2$$
$$= \sum_{p=1}^{+\infty} \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} \left|\left(\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} (f(x_1, x_2), e_p e^{i(k_1x_1+k_2x_2)}\right)dx_1\, dx_2\right|^2$$
$$= \sum_{p=1}^{+\infty} \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} \left|\left(\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(x_1, x_2)e^{-i(k_1x_1+k_2x_2)}\, dx_1 dx_2\, e_p\right)\right|^2$$
$$= \sum_{p=1}^{+\infty} \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} |(c_{k_1k_2}, e_p)|^2$$
$$= \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} (c_{k_1k_2}, c_{k_1k_2})$$

This theorem gives the Parseval's identity for vector valued functions in B_2 (H, K_2)

3. Pareseval Identity and B_2 $(C_{[0,1]}, (0, 2\pi))$.

Let $B_2(C_{[0,1]}, (0, 2\pi))$ be the space of square B integrable vector valued functions defined on $(0, 2\pi)$ to the supremum normed space of all complex continuous functions defined on the unit interval $[0, 1]$.

In the following, we show that Parseval's identity does not hold for a vector valued function in this $B_2(C_{[0,1]}, (0, 2\pi))$.

Take $f(x)=2+t\cos x,\ 0\leq x\leq 2\pi,\ 0\leq t\leq 1$

$f(x)\in B_2(C_{[0,1]}, (0, 2\pi))$

So

$$\|f(x)\| = \sup_{0\leq t\leq 1} |2+t\cos x| = \begin{cases} 2+\cos x & \cos x\geq 0 \\ 2 & \cos x<0 \end{cases}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \|f(x)\|^2\, dx = \frac{1}{2\pi}\int_0^{\pi/2} (2+\cos x)^2\, dx + \frac{1}{2\pi}\int_{\frac{3}{2}\pi}^{2\pi} (2+\cos x)^2\, dx = 4.25 \quad (2)$$

While Fourier's coefficients are:

$$c_0=2,\ c_1=\frac{t}{2},\ c_{-1}=\frac{t}{2}$$

$c_k=\theta$ for $k\geq 2,\ k\leq -2$ (θ is the zero vector of $C_{[0,1]}$)

So $\|c_0\|=2,\ \|c_1\|=\frac{1}{2},\ \|c_{-1}\|=\frac{1}{2}$

$\|c_k\|=0,\ k\geq 2,\ k\leq -2$

Then $$\sum_{k=-\infty}^{+\infty} \|c_k\|^2=4.5 \tag{3}$$

From (2) and (3), we get, Parseval's identity and Bessel's inequality are refuted in the case of square B integrable vector valued functions with values in a Banach space.

4. Parseval's identity and $B_2(m,\ (0,\ 2\pi))$

Consider the space of complex (or real) bounded sequences such as $(c_1,\ c_2,\ \cdots,\ c_n,\ \cdots)$ with the norm $\sup_i |c_i|$. This space is usually denoted by m. Let $B_2(m,\ (0,\ 2\pi))$ be the space of square B integrable vector valued functions defined on $(0,\ 2\pi)$ to m. Take

$$f(x)=\{\sin x,\ \frac{\sin 2x}{2},\ \cdots\cdots,\ \frac{\sin nx}{n},\ \cdots\cdots\},\ 0<x<2\pi$$

$$f(x)\in B_2(m,\ (0,\ 2\pi))$$

So $$\|f(x)\|=\sup_n \left|\frac{\sin nx}{n}\right|=|\sin x|.$$

$$\therefore\ \frac{1}{2\pi}\int_0^{2\pi} \|f(x)\|^2 dx=\frac{1}{2\pi}\int_0^{2\pi} \sin^2 x\, dx=\frac{1}{2} \tag{4}$$

Then its Fourier's coefficients are,

$$c_0=\frac{1}{2\pi}\int_0^{2\pi} f(x)\ dx=\theta \quad (\theta \text{ is the zero vector of } m).$$

$$c_k=\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}\, dx=\left\{0,\ 0,\ \cdots,\ 0,\ \cdots,\ \frac{-i}{2\pi}\int_0^{2\pi}\frac{\sin^2 kx}{k}\ dx,\ \cdots\cdots\right\}$$

$$=\left\{0,\ 0,\ \cdots,\ 0,\ \cdots,\ \frac{-i}{2k},\ \cdots\cdots\right\}$$

$$c_{-k}=\frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{ikx}\, dx=\left\{0,\ 0,\ \cdots,\ 0,\ \cdots,\ \frac{i}{2k},\ \cdots\cdots\right\}$$

$$\therefore\ \|c_0\|=0,\ \|c_k\|=\|c_{-k}\|=\frac{1}{2|k|}$$

$$\therefore\ \sum_{k=-\infty}^{+\infty} \|c_k\|^2=2\sum_{k=1}^{+\infty}\|c_k\|^2=\frac{1}{2}\sum_{k=1}^{+\infty}\frac{1}{k^2}=\frac{\pi^2}{12} \tag{5}$$

From (4) and (5), we get again the result: in the case $B_2(m, (0, 2))$ Parseval's identity and Bessel's inequality do not hold.

5. Non Hilbertization for a Banach space of values of square *B* integrable vector valued functions

It seems from the above discussions that Hilbertization of a Banach space is a necessary condition for the validity of Parseval's identity. But the question: is the condition sufficient?

We like to give here an example to show that Parseval's identity may be valid for a vector value function with values in non-Hilbert Banach space.

Let $$f(x)=\left\{x,\ \frac{x}{2},\ \cdots,\ \frac{x}{n},\ \cdots\cdots\right\},\quad 0<x<2\pi$$

$$f(x)\in B_2\ (m,\ (0,\ 2\pi)).$$

So

$$\frac{1}{2\pi}\int_0^{2\pi}\|f(x)\|^2dx=\frac{4}{3}\pi^2.$$

$$c_0=\left\{\pi,\ \frac{\pi}{2},\ \cdots,\ \frac{\pi}{n},\ \cdots\cdots\right\}$$

$$c_k=\frac{-i}{k}\left\{1,\ \frac{1}{2},\ \cdots,\ \frac{1}{n},\ \cdots\cdots\right\}$$

$$\|c_0\|=\pi,\quad \|c_k\|=\frac{1}{k},\quad \|c_{-k}\|=\frac{1}{k}$$

$$\therefore\ \sum_{k=-\infty}^{+\infty}\|c_k\|^2=\pi^2+2\sum_{k=1}^{+\infty}\|c_k\|^2=\frac{4}{3}\pi^2$$

As a result of this simple example, we can state that, Parseval's identity for a square *B* integrable vector valued function with values in a Banach space can not be considered as a criteria for Hilbertization of the space.

University of Assiut
Egypt

REFERENCES

[1] E. Hewitt & K. Stromberg, ***Real and abstract analysis***, Springer-Verlag, Berlin Heidelberg, (1975)

[2] E. Hille & R.S. Phillips, ***Functional analysis and semigroups***, Amer. Math. Soc.

Publ, Vol. 31, rev. ed, Amer. Math. Soc., Providence R.I., (1974).

[3] J.D. Pyrce, ***Basic methods of linear functional analysis***, Hutchinson University library, London, (1973)

[4] A. Zygmund, ***Trigonometric series***, Cambridge Univ. Press, New York, (1959)

[5] M.A. Kandil, ***The Weiner Paly theorem and other questions in the theory of abstract functions***, Math Sbornik Tom 73(115) No. 3 (1967), 306–319. English Translation, Math. USSR Sbornik Vol. 2, No. 3, (1967), 271–284.