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Unions of l-groups

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In this paper, lattice-ordered semigroups (l-semigroups) which are unions of groups are discussed. If the natural order on the idempotents is contained in the lattice order, such l-semigroups are shown to be unions of l-groups. Then l-bands are considered, and completely characterized in case the band is rectangular, right regular or commutative. An l-band of l-groups, the natural analogue in this context of a band of groups, is then defined and characterized. It is shown that such l-semigroups may be constructed from a lattice of l-homomorphic images of a given l-group.

Let S be a semigroup. We say that S is a *lattice-ordered semigroup* (or *l-semigroup*) if S admits a partial order $\leq$ which is a lattice order, such that

$$a(b\vee c)=ab\vee ac$$
$$(b\vee c)a=ba\vee ca,$$

and dually. Note that in [5] the statements about the meets are not part of the definition of l-semigroup.

Denote by $E(S)$ the set of idempotents of S. The relation $\tilde{\alpha}$ on $E(S)$ defined by $e\tilde{\alpha}f$ if and only if $ef=fe=e$ is the *natural partial order* on $E(S)$ [2]. It is easy to see that there exist l-semigroups S for which the orders $\leq$ and $\tilde{\alpha}$ on $E(S)$ are unrelated; just take any nonzero l-group G and adjoin a zero element greater than any element of G. However, as is shown in this paper, more natural examples of l-semigroups suggest that it is not unreasonable to require that the natural partial order on $E(S)$ be contained in the lattice order; that is, if $ef=fe=e$, then $e\leq f$. We shall call such l-semigroups *natural*, and shall confine our attention in this paper to them.

For the basic theory and notation for l-groups, the reader may refer to [1] or [3]. For semigroups, we shall rely on [2]. In particular, we shall denote by $\mathscr{R}$ and $\mathscr{H}$ Green's $\mathscr{R}$ and $\mathscr{H}$ relations. Also, if a semigroup S is a union of groups, it will be written as the disjoint union

$$S=\bigcup\{S_e : e\in E(S)\},$$

where each S_e is the maximal subgroup of S which contains e; here e is the identity for S_e. In fact, S_e is the $\mathscr{H}$ class of e ([2], p. 61).

The first theorem shows that it is possible to determine whether a natural l-semigroup is a union of l-groups by examining only its semigroup structure.

THEOREM 1. *A natural l-semigroup is a union of l-groups if and only if it is a union of groups.*

PROOF. The necessity is obvious. For the sufficiency, suppose that S is a natural l-semigroup and

$$S = \bigcup \{S_e : e \in E(S)\}$$

is a union of groups. We shall show that each $\mathscr{H}$ class S_e is an l-group. Since each S_e is clearly a partially ordered group, it suffices to show that $a \vee e \in S_e$, for each $a \in S_e$ (see [1], p. 18 or [3]). Now, $a \vee e \in S_g$, for some $g \in E(S)$. In fact, g is the smallest idempotent (according to the partial order $\tilde{\alpha}$) such that

$$g(a \vee e) = (a \vee e)g = a \vee e,$$

because if f is another such, then

$$fg = f(a \vee e)(a \vee e)^{-1} = g = (a \vee e)^{-1}(a \vee e)f = fg,$$

where $(a \vee e)^{-1}$ is the group inverse of $a \vee e$ in S_g. Since $e(a \vee e) = (a \vee e)e = a \vee e$, we have that $g \tilde{\alpha} e$. But let a^{-1} be the inverse of a in S_e. Then we claim that $e = (a^{-1} \wedge e)(a \vee e)$; this follows because

$$(a^{-1} \wedge e)(a \vee e) = (e \vee a^{-1}) \wedge (a \vee e) \geq e$$

and

$$(a^{-1} \wedge e)(a \vee e) = (e \wedge a) \vee (a^{-1} \wedge e) \leq e.$$

But then $ge = eg = e$ and so $e = g$. Thus S_e is an l-group.

We shall now examine those unions of l-groups for which there is some control over the multiplication and lattice operations. By an l-band we shall mean a natural l-semigroup which is a band (that is, each element is idempotent). An l-band of l-groups is a union of l-groups

$$S = \bigcup \{S_e : e \in E(S)\}$$

where $E(S)$ is an l-band, and for all $a \in S_e$ and $b \in S_f$, we have that $ab \in S_{ef}$, $a \wedge b \in S_{e \Delta f}$ and $a \vee b \in S_{e \nabla f}$. Here ∇ and Δ denote the lattice operations in $E(S)$; this is necessary since $E(S)$ need not be a sublattice of S. Note that requiring that $ab \in S_{ef}$ makes S a band of the groups S_e (see [2], p. 25).

We first look at l-bands themselves; the following theorem describes those which are rectangular, right regular and commutative. Recall that a band E is *rectangular* if it can be represented as $X\times Y$, where $(x_1, y_1)(x_2, y_2)=(x_1, y_2)$, and if $|X|=1$, then E is a *right zero semigroup* (see [2] p. 25). A band E is *right regular* if $efe=fe$, for all $e, f\in E$, (see [6]).

THEOREM 2. *Let E be a band.*

(a) If $E=X\times Y$ is a rectangular band, the following partial order makes E an l-band: equip X and Y with arbitrary lattice orders, and put the direct product of these orders on E. Conversely, any rectangular l-band is of this form. In particular, any lattice order on a right zero semigroup makes it an l-band.

(b) E is a right regular l-band if and only if and only if E is a natural lattice of right zero semigroups, each of which is a sublattice of E. That is, $E=\bigcup\{E_\alpha : \alpha\in\Lambda\}$, Λ is a lattice, and if $e\in E_\alpha$ and $f\in E_\beta$, then $e\vee f\in E_{\alpha\vee\beta}$. If E is the set of idempotents of a natural l-semigroup S, then E is a join-semilattice of S.

(c) E is a commutative l-band if and only if E is a lattice. In this case the multiplication and meet operation of the l-band coincide. If E is the set of idempotents of a natural l-semigroup S, then E is a sublattice of S and $e\wedge f=ef$.

PROOF. (a) Direct computation verifies that a rectangular band equipped with such a partial order becomes an l-band. Conversely, if $X\times Y$ is a rectangular l-band and $x_1, x_2\in X$, define $x_1\vee x_2$ to be a, where $(x_1, y)\vee(x_2, y)=(a, b)$ and define $x_1\wedge x_2$ similarly. It is easily checked that these definitions are well-defined and make X a lattice.

(b) Note that E is a right regular band if and only if it is a semilattice of its $\mathscr{R}$-classes (see [6]). First assume that E is a right regular l-band. We show that each $\mathscr{R}$-class is a sublattice. Now since E is natural, $ef\tilde{\alpha}f$, for all $e, f\in E$. Thus,

$$e\wedge f=(e\wedge f)^2=e\wedge ef\wedge fe\wedge f\leq fe\wedge ef\leq e\wedge f,$$

and so $e\wedge f=ef\wedge fe$. But then,

$$(e\wedge f)ef=ef\wedge fef=ef,$$
$$ef(e\wedge f)=fe\wedge ef=e\wedge f,$$

and thus $(e\wedge f)\ \mathscr{R}\ ef$. Also, it is easily verified that if $e\mathscr{R}f$, then $(e\vee f)\mathscr{R}e$. Thus, each $\mathscr{R}$-class is clearly a sublattice. We now show that the index set Λ is a lattice. But since $\mathscr{R}$ is a congruence relation on a right regular band, and $(e\wedge f)\mathscr{R}ef$, we have that Λ is a meet semilattice. Now suppose that $e\mathscr{R}g$ and $f\mathscr{R}h$. The following then holds, because E is natural, and an idempotent in an $\mathscr{R}$-class is a left identity:

$$(e\vee f)(g\vee h)=eg\vee eh\vee fg\vee fh=$$
$$(g\vee h)\vee(eh\vee fg)=g\vee h.$$

Therefore, $(e\vee f)\ \mathscr{R}\ (g\vee h)$ and so Λ is a lattice. Conversely, if E is a lattice of right zero semigroups as specified in the statement of the theorem, it will follow immediately that E is a right regular l-band if E is natural. But if $ef=fe=e$, then $e\wedge f\in\mathscr{R}_{ef}=\mathscr{R}_e$ and

$$e\wedge f=e(e\wedge f)=e\wedge ef=e.$$

Finally, if $E=E(S)$ where S is a natural l-semigroup, we of course have that

$$e\vee f\leq e\nabla f$$

where $\vee$ is the join in S and ∇ is the join in E. However, $e\vee f$ is an idempotent, because

$$(e\vee f)^2=e\vee ef\vee fe\vee f=e\vee f$$

and so $$e\vee f=e\nabla f.$$

(c) This follows almost immediately from (b), because the right zero semigroups there reduce to singletons. The last statement holds, because

$$ef\geq(e\wedge f)^2=e\wedge ef\wedge fe\wedge f=ef.$$

Notice, of course, that (b) and (c) provide information about $\mathscr{L}$-unipotent and inverse semigroups which are l-bands of l-groups, since the sets of idempotents of these types of semigroups are respectively right regular or commutative bands.

We now characterize l-bands of l-groups.

THEOREM 3. *Let S be a natural l-semigroup. Then S is an l-band of l-groups if and only if*

(1) $S=\bigcup\{S_e : e\in E(S)\}$ *is a band of groups, and*

(2) *given $a\in S_e$ and $b\in S_f$, $a\leq b$ if and only if $e\leq f$ and $a\leq ebe\in S_e$.*

PROOF. ($\Longrightarrow$) (1) is clear. If $a\leq b$, then $a\vee b=b\in S_f$, and so, by the definition of l-band of l-groups, $e\vee f\in S_f$; thus, $e\leq e\nabla f=f$. Clearly, $a=eae\leq ebe$, and $ebe\in S_{efe}$. But

$$e=e^3\leq efe\leq e$$

and so $efe=e$. Conversely, $a\leq ebe\leq fbf\leq b$. ($\Longleftarrow$) We first note that (2) implies that each S_e is convex, i.e., if $a\leq x\leq b$ and $a, b\in S_e$, then $x\in S_e$. Now, by Theorem 1, S is a band of l-groups. We now must show that $E(S)$ is an *l-band*. If $e, f\in E(S)$ and $e\vee f\in S_g$, since $e\leq e\vee f$, we have by (2) that $e\leq g$. Thus, $e\vee$

$f \leq g$. Define $e \nabla f = g$. Similarly, if $e \wedge f \in S_h$, let $e \Delta f = h$. From (2) it is easy to check that $e \Delta f$ and $e \nabla f$ are respectively the g. l. b. and l. u. b. of $E(S)$ with the order it inherits from S. Now,

$$e(f \nabla g) \geq ef \nabla eg \geq ef \vee eg = e(f \vee g).$$

But because we have a band of groups, $e(f \vee g) \in S_{e(f \nabla g)}$. Since $S_{e(f \nabla g)}$ is convex and each group contains only one idempotent, $e(f \nabla g) = ef \nabla eg$. The dual proof (and the proof for distributivity on the right) is similar. Now, if $a \in S_e$ and $b \in S_f$, we must show that $a \vee b \in S_{e \nabla f}$. Suppose that $a \vee b \in S_g$. Then $e \leq g$ and $f \leq g$ and so $e \nabla f \leq g$. But then

$$e \nabla f = (e \nabla f)^3 \leq (e \nabla f) g (e \nabla f) \tilde{\alpha} e \nabla f$$

and so $(e \nabla f) g (e \nabla f) = e \nabla f$. Now

$$a \leq a \vee af \vee fa \vee faf = (e \vee f) a (e \vee f) \leq$$
$$(e \nabla f)(a \vee b)(e \nabla f) \leq g(a \vee b) g = a \vee b$$

and similarly for b. Thus

$$a \vee b \leq (e \nabla f)(a \vee b)(e \nabla f) \leq a \vee b$$

and so

$$a \vee b = (e \nabla f)(a \vee b)(e \nabla f) \in S_{(e \nabla f) g (e \nabla f)} = S_{e \nabla f}.$$

The proof for meets is similar.

We remark that in the case where S is an orthodox semigroup (1) is equivalent to

(1′) S is equal to the kernel of its greatest idempotent separating congruence, i. e., for each $x \in S$, there exists an inverse x' of x such that $x'xex'x = x'ex$, for all $e \in E(S)$ [4]. When S is inverse, this means simply that each element commutes with every idempotent. We also remark that we could replace (2) by

(2′) given $a \in S_e$ and $b \in S_f$ with $a \leq b$, then $e \leq f$. Statement (2) above emphasizes that the order on S is completely determined by the order on $E(S)$ and the orders on the l-groups S_e.

We can in fact explicitly describe the structure of an l-band of l-groups; this will enable us to construct examples of such semigroups. Let $S = \cup \{S_e : e \in E(S)\}$ be an l-band of l-groups, and let

$$H = \prod \{S_e : e \in E(S)\},$$

the unrestricted cardinal product of the l-groups S_e; that is,

$$H=\{a\colon E(S)\longrightarrow \bigcup S_e\colon a(e)\in S_e\}.$$

and $a\geq b$ if and only if $a(e)\geq b(e)$ for all e. Now define G to be

$$\{a\in H\colon e[a(f)]e=a(e),\ \text{whenever } e<f,\ e,f\in E(S)\}.$$

Note that $e[a(f)]e$ is necessarily an element of S_e, an so this definition makes sense. Because the group and lattice operations in H are pointwise, G is clearly an l-subgroup of H.

We now show that each of the l-groups S_e is an l-homomorphic image of G. For consider

$$\phi_e\colon G\longrightarrow S_e\colon a\longrightarrow a(e).$$

This is clearly an l-homomorphism. We need only check that ϕ_e is onto. Let

$$s\in(S_e)^+=\{t\in S_e\colon t\geq e\};$$

Since $(S_e)^+$ generates S_e as a group [1], it is sufficient to show that s is in the image of ϕ_e. Define $a\in H$ by letting $a(f)=fsf\vee f$. Now, $fsf\in S_{fef}$ and $fef\ \tilde{\alpha}\ f$, and so $a(f)\in S_f$. Clearly $a(e)=s$; thus it remains only to show that $a\in G$. For this purpose, suppose that $f,\ g\in E(S)$ and $f<g$. Then

$$f[a(g)]f=f(gsg\vee g)f=fgsgf\vee fgf=fgsgf\vee f=fsf\vee f,$$

since $s\geq gsg\geq fsf$, and so $fgsgf=fsf$. Thus, each l-group S_e is an element of $H(G)$, the lattice of l-homomopphic images of $G(H(G))$ is just the antilattice of $\mathcal{L}(G)$, the lattice of l-ideals of G, that it, the kernels of l-homomorphisms; see [3]). In fact, we may define $\phi\colon E(S)\longrightarrow H(G)$ by $\phi(e)=S_e$. If $e<f$, than $\phi(e)$ is an l-homorphic image of $\phi(f)$ via the map $a\longrightarrow eae$, and so the map ϕ is in fact order-preserving. The order and multiplication on S may then be defined as follows. Let

$$a_1=K_1g_1\in G/K_1=\phi(e_1)\ \text{and}\ a_2=K_2g_2\in G/K_2=\phi(e_2).$$

Then

$$a_1a_2=K_3g_1g_2\in G/K_3=\phi(e_1e_2)\ \text{and}\ a_1\leq a_2\ \text{iff}\ K_1\supseteq K_2\ \text{and}\ K_1g_1\leq K_1g_2.$$

Conversely, we may use the ideas above to construct examples of l-bands of l-groups. Let G be an l-group, E an l-band, and L a subset of $H(G)$ with $\phi\colon E\longrightarrow L$ an order-preserving map onto L. For each $e\in E$, choose a distinct copy of the l-group $\phi(e)$, and let S be the disjoint union of all such l-groups. Then we may define multiplication and order as in (*) above. It is easy to check that S is a band of groups, and that the order on E agrees with the order defined on S. From Theorem 3, it remains only to show that S is an

l-semigroup. But S is clearly a lattice of lattices, and so a lattice. That multiplication distributes over the lattice operations follows because this is true in both E and in each $\phi(e)$.

We will conclude with two simple examples of unions of l-groups which are not l-bands of l-groups. Let $S=\boldsymbol{R}^{+}\times\{0,1\}$ equipped with the following total order:

$$(r,\alpha)<(s,\beta) \text{ iff } r<s \text{ or } r=s \text{ and } \alpha<\beta.$$

Define multiplication on S by letting

$$(r,\alpha)\cdot(s,\beta)=(rs,\alpha\beta).$$

Then S is a band of the groups $\boldsymbol{R}^{+}\times\{0\}$ and $\boldsymbol{R}^{+}\times\{1\}$ and is a natural l-semigroup with idempotents $(1,0)$ and $(1,1)$. However, condition (2) of Theorem 3 makes it obvious that S is not an l-band of l-groups. Consider the ordered set S again with the following multiplication:

$$(r,\alpha)\cdot(s,\beta)=\begin{cases}(rs,\ 0) & \text{if } rs<1\\ (rs,\ \alpha\beta) & \text{if } rs\geq 1\end{cases}$$

Then S is a union of the l-groups $\boldsymbol{R}^{+}\times\{0\}$ and $\boldsymbol{R}^{+}\times\{1\}$ but is not a band of groups.

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