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# ON EIGEN-FORMS ON SURFACES WITH NULL GAUSSIAN CURVATURE IN ELLIPTIC SPACE $S_3$

By M. A. Soliman

## 1. Introduction

Let $(F, g)$ be an oriented compact connected $n$-dimensional Riemannian manifold. To each $p$-form $\omega$ on $F$, there is associated the $(p+1)$ —form $d\omega$ and the $(n-p)$— form $*\omega$ respectively, $*$ being the Hodge operator.

The exterion codifferential $\delta$ is then defined by

$$\delta\omega=(-1)^p *^{-1}D*\omega, \tag{1}$$

$*^{-1}$ being the inverse mapping to $*$([2], [3], [4]). The Laplacian $\Delta$ on $p$-forms is given by

$$\Delta\omega=(D\delta+\delta D)\omega. \tag{2}$$

We say that $\lambda\in R$ belongs to spec$^{(p)}$ $(\Delta)$ if there is a nontrivial $p$-form $\omega$ on $F$ such that

$$\Delta\omega=\lambda\omega. \tag{3}$$

The general problem is to exhibit spec$^{(p)}$ $(\Delta)$ for a given $(F, g)$. Up to now, little is known. Spec$^{(o)}$ $(\Delta)$ is known just for the hypersphere. Recently [5] it has been proved that for a unit sphere of the Euclidean space $E^3$ spec$^{(1)}$ $(\Delta)$ equal to 2. There are no general methods for solving the general problem. We are going to use the stokes theorem

$$\int_F D\phi=0 \tag{4}$$

where $\phi$ is an $(n-1)$ form.

Consider an 3-dimensional projective space $P_3$ refered to a moving frame $\{A_i\}$ of four linearly independent analytic points $A_1$, $A_2$, $A_3$, $A_4$. An infinitesimal displacement of such a frame is determined by the equations.

$$dA_i=\omega_i^j\ A_j,\ \ (i,\ j,\ k=1,\ 2,\ 3,\ 4) \tag{5}$$

where the one-forms $\omega_i^j$ (Pfaff's differental forms) are invariant one-forms of the projective group $PG(3, R)$ whose structural equations have the form

$$D\omega_i^j=\omega_i^k\wedge\omega_k^j. \tag{6}$$

A homogeneous space $S_3=(P_3, H_1^3)$ is called an *elliptic space* if $H_1^3$ is a subgroup of the group $PG(3, R)$, the transformations in the subgroup $H_1^3$ do not move a non-degenerate imaginary quadric (absolute) $\sigma$. We choose a moving frame conjugate to any arbitrary manifold embedded in $S_3$ as a normalized polar tetrahedron $\{A_i\}$. In such moving frame, the absolute $\sigma$ is determined by the equation

$$\sum_{i=1}^{4}(x^i)^2=0. \tag{7}$$

The conditions of the stationary subgroup $H_1^3$ are

$$\omega_i^i=0,\ \omega_i^j+\omega_j^i=0 \tag{8}$$

## 2. Linear forms on surface

Let $F$ be a closed surface with null Gaussian Curvature. We are going to investigate its coordinate neighbourhood $U\subset F$. To each point $A_1\in F$, let us associate a moving normalized polar tetrahedron $\{A_i\}$ such that the points $A_2$, $A_3$ are in the tangent plane to the surface $F$ at the point $A_1$.

The fundamental equations of a moving tetrahedron are:

$$\begin{aligned} dA_1&=\omega_1^2 A_2+\omega_1^3 A_3,\ \omega_1^4=0,\\ dA_2&=\omega_2^1 A_1+\omega_2^3 A_3+\omega_2^4 A_4,\\ dA_3&=\omega_3^1 A_1+\omega_3^2 A_2+\omega_3^4 A_4,\\ dA_4&=\omega_3^2 A_2+\omega_4^3 A_3. \end{aligned} \tag{9}$$

The differential equation of the surface $F$ in the first differential neighbourhood is

$$\omega_1^4=0. \tag{10}$$

Exterior differentiation and using Cartan's Lemma [4] we get,

$$\begin{aligned} \omega_2^4&=\alpha\omega_1^2+\beta\omega_1^3,\\ \omega_3^4&=\beta\omega_1^2+\gamma\omega_1^3. \end{aligned} \tag{11}$$

The Gaussian curvature of the surface $F$ is given by

$$K=\frac{D\omega_3^2}{\omega_1^2\wedge\omega_1^3}=\frac{\omega_3^1\wedge\omega_1^2+\omega_3^4\wedge\omega_4^2}{\omega_1^2\wedge\omega_1^3}$$

$$=1+\alpha\gamma-\beta^2. \tag{12}$$

Hence the differential equations of the surface $F$ in the second differential neighbourhood are

$$\begin{aligned}\omega_1^4&=0,\\ \omega_2^4&=\alpha\omega_1^2+\beta\omega_1^3,\\ \omega_3^4&=\beta\omega_1^2+\gamma\omega_1^3,\end{aligned} \tag{13}$$

with

$$1+\alpha\gamma-\beta^2=0.$$

The purpose of this work is to prove the following

THEOREM. *Let $(F, g)$ be a closed surface with null Gaussian curvature in elliptic space $S_3$, $g$ being the induced metric. Let $\lambda\in spec^{(1)}(\Delta)$. Then the most general eigenvalue satisfying $\Delta\omega=\lambda\omega$,* is that $\lambda=0$.

PROOF. On the surface $F$, be given a 1-form $\omega$ in $U$.

$$\omega=a\omega_1^2+b\omega_1^3, \tag{14}$$

$a, b+: U\longrightarrow R$ being functions. They are defined by

$$\begin{aligned}da-b\omega_2^3&=a_1\omega_1^2+a_2\omega_1^3,\\ db+a\omega_2^3&=b_1\omega_1^2+b_2\omega_1^3.\end{aligned} \tag{15}$$

The exterior differentiation implies

$$\left\{da_1-(a_2+b_1)\omega_2^3\right\}\wedge\omega_1^2+\left\{da_2+(a_1-b_2)\omega_2^3\right\}\wedge\omega_1^3=0. \tag{16}$$

$$\left\{db_1+(a_1-b_2)\omega_2^3\right\}\wedge\omega_1^2+\left\{db_2+(a_2+b_1)\omega_2^3\right\}\wedge\omega_1^3=0.$$

Applying here Cartan's lemma we get the functions.

$a_{ij}, b_{kj}: U\longrightarrow R$ such that

$$\begin{aligned}da_1-(a_2+b_1)\omega_2^3&=a_{11}\omega_1^2+a_{12}\omega_1^3,\\ da_2+(a_1-b_2)\omega_2^3&=a_{12}\omega_1^2+a_{22}\omega_1^3,\\ db_1+(a_1-b_2)\omega_2^3&=b_{11}\omega_1^2+b_{12}\omega_1^3,\\ db_2+(a_2+b_1)\omega_2^3&=b_{12}\omega_1^2+b_{22}\omega_1^3.\end{aligned} \tag{17}$$

The consequences of exterior differentiation of (17) are

$$\{da_{11}-(2a_{12}+b_{11})\omega_2^3\}\wedge\omega_1^2+\{da_{12}+(a_{11}-a_{22}-b_{12})\omega_2^3\}\wedge\omega_1^3=0,$$
$$\{da_{12}+(a_{11}-a_{22}-b_{22})\omega_2^3\}\wedge\omega_1^2+\{da_{22}+(2a_{12}-b_{22})\omega_2^3\}\wedge\omega_1^3=0,$$
$$\{db_{11}+(a_{11}-2b_{12})\omega_2^3\}\wedge\omega_1^2+\{db_{12}+(b_{12}+b_{11}-b_{22})\omega_1^2\}\wedge\omega_1^3=0,$$
$$\{db_{12}+(a_{12}+b_{11}-b_{22})\omega_2^3\}\wedge\omega_1^2+\{db_{22}+(a_{22}+2b_{12})\omega_2^3\}\wedge\omega_1^3=0. \tag{18}$$

Using Cartan's lemma equations (18) give the existence of functions $A_i,\ B_i:=U\longrightarrow R$ such that

$$\begin{aligned}
&da_{11}-(2a_{12}+b_{11})\omega_2^3=A_1\omega_1^2+A_2\omega_1^3,\\
&da_{12}+(a_{11}-a_{22}-b_{12})\omega_2^3=A_2\omega_1^2+A_3\omega_1^3,\\
&da_{22}+(2a_{12}-b_{22})\omega_2^3=A_3\omega_1^2+A_4\omega_1^3,\\
&db_{11}+(a_{11}-2b_{12})\omega_2^3=B_1\omega_1^2+B_2\omega_1^3,\\
&db_{12}+(a_{12}+b_{11}-b_{22})\omega_2^3=B_2\omega_1^2+B_3\omega_1^3,\\
&db_{22}+(a_{22}+2b_{12})\omega_2^3=B_3\omega_1^2+B_4\omega_1^3.
\end{aligned} \tag{19}$$

Now for 1-form, we have

$$*(p\omega_1^2+q\omega_1^3)=-q\omega_1^2+p\omega_1^3 \tag{19}$$

$$*^{-1}(p\omega_1^2+q\omega_1^3)=q\omega_1^2-p\omega_1^3. \tag{20}$$

$$\Delta\omega=(*^{-1}D*D-D*^{-1}D*)\omega. \tag{21}$$

In our case we have

$$\begin{aligned}
&\omega=a\omega_1^2+b\omega_1^3,\\
&D\omega=(b_1-a_2)\omega_1^2\wedge\omega_1^3,\\
&*D\omega=b_1-a_2,\\
&D*D\omega=(b_{11}-a_{12})\omega_1^2+(b_{12}-a_{22})\omega_1^3,\\
&*^{-1}D*D\omega=(b_{12}-a_{22})\omega_1^2-(b_{11}-a_{12})\omega_1^2,\\
&*\omega=-b\omega_1^2+a\omega_1^3\\
&D*\omega=(b_2+a_1)\omega_1^2\wedge\omega_1^3\\
&*^{-1}D^*\omega=b_2+a_1
\end{aligned} \tag{22}$$

$$D*^{-1}D*\omega=(a_{11}+b_{12})\omega_1^2+(a_{12}+b_{22})\omega_1^3. \tag{23}$$

Hence for the 1-form $\omega$ the Laplacian

$$\Delta\omega=-(a_{11}+a_{22})\omega_1^2-(b_{11}+b_{22})\omega_1^3. \tag{24}$$

If the form $\omega$ satisfying (3), then

$$a_{11}+a_{22}=-\lambda a,\ \ b_{11}+b_{22}=-\lambda b. \tag{25}$$

Because of equations (19) we get

$$\begin{aligned} &A_1+A_3=-\lambda a_1,\ \ A_2+A_4=-\lambda a_2,\\ &B_1+B_3=-\lambda b_1,\ \ B_2+B_4=-\lambda b_2. \end{aligned} \tag{26}$$

For a general form $\omega$ we can get

$$\begin{aligned} D*D\{(a_1-b_2)^2+(a_2+b_1)^2\}&=2\{(a_{11}-b_{12})^2+(a_{12}-b_{22})^2+\\ &+(a_{12}+b_{11})^2+(a_{22}+b_{12})^2\}\omega_1^2\wedge\omega_1^3\\ &+(-2\lambda)\{(a_1-b_2)^2+(a_2+b_1)^2\}\omega_1^2\wedge\omega_1^3. \end{aligned}$$

Using the stockes theorem on $D*D\omega$, we get

$$a_1-b_2=0,\ \ a_2+b_1=0.$$

$$a_{11}-b_{12}=a_{12}-b_{22}=a_{12}+b_{11}=a_{22}+b_{12}=0.$$

From which follows that

$$a_{11}+a_{22}=0,\ \ b_{11}+b_{22}=0. \tag{27}$$

Comparing (27) with (25) it follows directly that

$$\lambda=0.$$

This proves our theorem.

King Abdulaziz University
Jeddah, Saudi Arabia

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