Kyungpook Math. J.
Volume 23, Number 1
June, 1983

CHARACTERIZATIONS OF UNIFORMLY CONTINUOUS AND CAUCHY-REGULAR FUNCTIONS USING NETS

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A function from a uniform space to a uniform space is a *Cauchy-regular (C-regular) function* if it preserves Cauchy filterbases. The class of C-regular functions is (strictly included) between the class of uniformly continuous functions and the class of continuous functions. A previous paper (see [3]) presents some basic properties of C-regular functions, demonstrates that some of the most useful theorems about uniformly continuous functions also hold for C-regular functions, and shows that many functions which occur in analysis are C-regular but not uniformly continuous.

The purpose of this note is to characterize C-regular functions in terms of nets and to give, for the sake of comparison, net characterizations of continuous and uniformly continuous functions. Examples are given to illustrate the usefulness of the net characterization of uniform continuity.

1. Classification of nets in a uniform space

Let $(X, \mathscr{U})$ be a uniform space (see [2]). A net (x_δ) in X, with directed set $(D, \geq)$, is *convergent* if there is a point a in X such that: for each set U in $\mathscr{U}$, there exists an element δ_U in D such that

$$\delta \in D \text{ and } \delta \geq \delta_U \Longrightarrow (x_\delta, a) \in U.$$

We also say (x_δ) *converges to* a, and write $(x_\delta) \longrightarrow a$. A net (x_δ) in X is *Cauchy* if: for each set U in $\mathscr{U}$, there exists an element δ_U in D such that

$$\delta_1, \delta_2 \in D \text{ and } \delta_1, \delta_2 \geq \delta_U \Longrightarrow (x_{\delta_1}, x_{\delta_2}) \in U.$$

Two nets (x_δ) and (y_δ) in X are *parallel*, written $(x_\delta) \| (y_\delta)$, if they have the same directed set $(D, \geq)$ and if: for each set U in $\mathscr{U}$, there exists an element δ_U in D such that

$$\delta \in D \text{ and } \delta \geq \delta_U \Longrightarrow (x_\delta, y_\delta) \in U.$$

Finally, two nets (x_δ) and (y_δ) in X are *equivalent*, written $(x_\delta) \simeq (y_\delta)$, if they have the same directed set $(D, \geq)$ and if: for each set U in $\mathscr{U}$, there exists an element δ_U in D such that

$$\delta_1,\ \delta_2 \in D \text{ and } \delta_1,\ \delta_2 \geq \delta_U \Longrightarrow (x_{\delta_1},\ y_{\delta_2}) \in U.$$

As an immediate consequence of these definitions we have the following remark which indicates how parallel and equivalent nets are related.

REMARK. Let (x_δ) and (y_δ) be nets in X with the same directed set $(D, \geq)$. Let $a \in X$. Then:

(1) $(x_\delta) \longrightarrow a \Longrightarrow (x_\delta)$ is Cauchy.

(2) $(x_\delta) \| (x_\delta)$.

(3) (x_δ) is Cauchy $\Longleftrightarrow (x_\delta) \simeq (x_\delta)$.

(4) $(x_\delta) \longrightarrow a$ and $(y_\delta) \longrightarrow a \Longrightarrow (x_\delta) \simeq (y_\delta)$.

(5) $(x_\delta) \simeq (y_\delta) \Longrightarrow (x_\delta) \| (y_\delta)$.

(6) $(x_\delta) \simeq (y_\delta) \Longrightarrow (x_\delta)$ is Cauchy.

(7) (x_δ) and (y_δ) are Cauchy and $(x_\delta) \| (y_\delta) \Longleftrightarrow (x_\delta) \simeq (y_\delta)$.

(8) $(x_\delta) \longrightarrow a$ and $(x_\delta) \| (y_\delta) \Longrightarrow (y_\delta) \longrightarrow a$.

(9) (x_δ) is Cauchy and $(x_\delta) \| (y_\delta) \Longrightarrow (y_\delta)$ is Cauchy.

(10) $(x_\delta) \| (y_\delta) \Longleftrightarrow (y_\delta) \| (x_\delta)$.

(11) $(x_\delta) \simeq (y_\delta) \Longleftrightarrow (y_\delta) \simeq (x_\delta)$.

2. Net characterizations of continuous, C-regular, and uniformly continuous functions

The criterion for continuity in terms of nets can be stated as follows (see [2], p. 86; and [1]).

THEOREM 1. *Let $(X, \mathscr{T}_x)$ and $(Y, \mathscr{T}_y)$ be topological spaces, and let f: $X \longrightarrow Y$ be a function from X to Y. Then the following are equivalent:*

(1) *f is continuous.*

(2) *f preserves convergent nets and their limits, i.e., if (x_δ) is a net in X and $x \in X$ such that $(x_\delta) \longrightarrow x$, then $(f(x_\delta)) \longrightarrow f(x)$. When $(Y, \mathscr{T}_y)$ is a T_1-space,*

(1) *and* (2) *are each equivalent to:*

(3) *f preserves convergent nets, i.e., if (x_δ) is a convergent net in X, then $(f(x_\delta))$ is a convergent net in Y.*

Before characterizing C-regular functions in terms of nets, we recall several definitions. Let $(X, \mathscr{U})$ be a uniform space. A filterbase $\mathscr{N}$ in X is *Cauchy* if: for each set U in $\mathscr{U}$, there exists a set N in $\mathscr{N}$ such that $N \times N \subseteq U$. Two filterbases $\mathscr{N}$ and $\mathscr{M}$ in X are *equivalent*, written $\mathscr{N} \sim \mathscr{M}$, if: for every set U in $\mathscr{U}$, there exists sets N in $\mathscr{N}$ and M in $\mathscr{M}$ such that $N \times M \subseteq U$.

THEOREM 2. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. Let $f: X \longrightarrow Y$ be a function from X into Y. Then the following are equivalent:*

(1) *f is C-regular (f preserves Cauchy filterbases), i.e., if $\mathscr{N}$ is a Cauchy filterbase in X, then the filterbase $f[\mathscr{N}] = \{f[N] : N \in \mathscr{N}\}$, where $f[N] = \{f(x) : x \in N\}$, in Y is Cauchy.*

(2) *f preserves equivalent filterbases, i.e., if $\mathscr{N}$ and $\mathscr{M}$ are equivalent filterbases in X, then $f[\mathscr{N}]$ and $f[\mathscr{M}]$ are equivalent filterbases in Y.*

(3) *f preserves Cauchy nets, i.e., if (x_δ) is a Cauchy net in X, then $(f(x_\delta))$ is a Cauchy net in Y.*

(4) *f preserves equivalent nets, i.e., if (x_δ) and (y_δ) are equivalent nets in X, then $(f(x_\delta))$ and $(f(y_\delta))$ are equivalent nets in Y.*

PROOF. We shall sketch the proof that $(1) \Longrightarrow (2) \Longrightarrow (4) \Longrightarrow (3) \Longrightarrow (1)$. Assume (1). Let $\mathscr{N}$ and $\mathscr{M}$ be filterbases in X such that $\mathscr{N} \sim \mathscr{M}$. Then $\mathscr{N} \wedge \mathscr{M} = \{N \cup M : N \in \mathscr{N} \text{ and } M \in \mathscr{M}\}$ is a Cauchy filterbase in X. By (1), the set $f[\mathscr{N} \wedge \mathscr{M}] = f[\{N \cup M : N \in \mathscr{N} \text{ and } M \in \mathscr{M}\}] = \{f[N \cup M] : N \in \mathscr{N} \text{ and } M \in \mathscr{M}\} = \{f[N] \cup f[M] : N \in \mathscr{N} \text{ and } M \in \mathscr{M}\} = f[\mathscr{N}] \wedge f[\mathscr{M}]$ is a Cauchy filterbase in Y. It follows that $f[\mathscr{N}]$ and $f[\mathscr{M}]$ are filterbases in Y with $f[\mathscr{N}] \sim f[\mathscr{M}]$. This proves that $(1) \Longrightarrow (2)$.

Assume (2). Let (x_δ) and (y_δ) be equivalent nets in X each having the directed set $(D, \geq)$. Consider the filterbases in X generated by these nets:

$$\mathscr{N} = \{\{x_\delta : \delta \in D \text{ and } \delta \geq \delta_0\} : \delta_0 \in D\}$$

$$\mathscr{M} = \{\{y_\delta : \delta \in D \text{ and } \delta \geq \delta_0\} : \delta_0 \in D\}.$$

Since $(x_\delta) \simeq (y_\delta)$, we have $\mathscr{N} \sim \mathscr{M}$. By (2), the filterbases

$$f[\mathscr{N}] = \{\{f(x_\delta) : \delta \in D \text{ and } \delta \geq \delta_0\} : \delta_0 \in D\}$$

$$f[\mathscr{M}] = \{\{f(y_\delta) : \delta \in D \text{ and } \delta \geq \delta_0\} : \delta_0 \in D\}$$

in Y are equivalent. It follows that $(f(x_\delta)) \simeq (f(y_\delta))$. This proves that $(2) \Longrightarrow (4)$.

Assume (4). Let (x_δ) be a Cauchy net in X. Then $(x_\delta) \simeq (x_\delta)$. By (4), we have $(f(x_\delta)) \simeq (f(x_\delta))$ whence $(f(x_\delta))$ is a Cauchy net in Y. Thus $(4) \Longrightarrow (3)$.

Finally, assume (3). Let $\mathscr{N}$ be a Cauchy filterbase in X. Define the set

$$D(\mathscr{N})=\{(x, N): x\in N \text{ and } N\in\mathscr{N}\},$$

define the binary relation $\geq$ on $D(\mathscr{N})$ by

$$(x_2, N_2)\geq(x_1, N_1)\Longleftrightarrow N_2\subseteq N_1;$$

and define the function $g_{\mathscr{N}}: D(\mathscr{N})\longrightarrow X$ by the correspondence $g_{\mathscr{N}}(x, N)=x$. Then $(g_{\mathscr{N}}, \geq)$ is a net in X. Since $\mathscr{N}$ is Cauchy, $(g_{\mathscr{N}}, \geq)$ is Cauchy. By (3), the net $(f\circ g_{\mathscr{N}}, \geq)$ in Y is Cauchy. It follows that the filterbase $f[\mathscr{N}]$ is Cauchy. Thus (3)$\Longrightarrow$(1).

THEOREM 3. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. Let $f: X\longrightarrow Y$ be a function from X into Y. Then the following are equivalent*:

(1) *f is uniformly continuous, i.e., for every set V in $\mathscr{V}$, there exists a set U in $\mathscr{U}$ such that*: $(x, y)\in U\Longrightarrow(f(x), f(y))\in V$.

(2) *f preserves parallel nets, i.e., if (x_σ) and $(y_{\bar{\delta}})$ are parallel nets in X, then $(f(x_{\bar{\delta}}))$ and $(f(y_\delta))$ are parallel nets in Y.*

PROOF. First, we prove that (1)$\Longrightarrow$(2). Assume (1). Let $(x_{\bar{\delta}})$ and(y^{δ}) be the parallel nets in X each having the directed set $(D, \geq)$. We must prove that nets $(f(x_{\bar{\delta}}))$ and $(f(y_{\bar{\delta}}))$ are parallel. Of course, they have the same directed set$(D, \geq)$. Let $V\in\mathscr{V}$. Let $U\in\mathscr{U}$ as given by (1). Since $(x_{\bar{\delta}})\|(y_{\bar{\delta}})$, there exists an element δ_U in D such that: $\delta\in D$ and $\delta\geq\delta_U\Longrightarrow(x_{\bar{\delta}}, y_{\bar{\delta}})\in U$. But by our characterization of U, we have: $\delta\in D$ and $\delta\geq\delta_U\Longrightarrow(f(x_{\bar{\delta}}), f(y_{\bar{\delta}}))\in V$. Therefore, $(f(x_{\bar{\delta}}))\|(f(y_{\bar{\delta}}))$. Thus (2) holds.

Suppose (1) is false. Then there exists a set V in $\mathscr{V}$ such that: for every set U in $\mathscr{U}$, there exists an ordered pair (x_U, y_U) in U with $(f(x_U), f(y_U))\notin V$. Since $\mathscr{U}$ is a filterbase, $(\mathscr{U}, \subseteq)$ is a directed set. Consequently, $(x_U: U\in\mathscr{U})$ and $(y_U: U\in\mathscr{U})$ are nets in X. Moreover, $(x_U)\|(y_U)$ since (x_U) and (y_U) have the same directed set $(\mathscr{U}, \subseteq)$ and: for each set U' in $\mathscr{U}$, there exists an element U in $\mathscr{U}$ such that

$$U'\in\mathscr{U} \text{ and } U'\subseteq U\Longrightarrow(x_{U'}, y_{U'})\in U.$$

On the other hand, $(f(x_U): U\in\mathscr{U})$ and $(f(y_U): U\in\mathscr{U})$ are not parallel nets in Y since: there exists a set V in $\mathscr{V}$ such that for every element U in $\mathscr{U}$ there exists an element U in $\mathscr{U}$ such that $U\subseteq U$ and $(f(x_U), f(y_U))\notin V$. Thus (2) is false. This proves that (2)$\Longrightarrow$(1).

In summary, continuous functions are precisely those functions which preserve convergent nets; C-regular functions are precisely those functions which preserve Cauchy nets (or equivalent nets); and uniformly continuous functions are precisely those functions which preserve parallel nets.

3. Examples

Theorem 3 is useful in determining whether or not functions are uniformly continuous. As an illustration, we show that a linear transformation $f: X \longrightarrow Y$ from a topological vector space $(X, \mathscr{T}_x)$ to a topological vector space $(Y, \mathscr{T}_y)$, each over the real or complex field K, is uniformly continuous if it is continuous at 0 in X. Note that two nets (x_δ) and (y_δ) in X are parallel if and only if: they have the same directed set, and $(x_\delta - y_\delta) \longrightarrow 0$. Let (x_δ) and (y_δ) be parallel nets in X. Then $(x_\delta - y_\delta) \longrightarrow 0$. Since f is continuous at 0 in X, we have $(f(x_\delta - y_\delta)) \longrightarrow f(0)$. Since f is a linear transformation, this becomes $(f(x_\delta) - f(y_\delta)) \longrightarrow 0$ whence $(f(x_\delta))$ and $(f(y_\delta))$ are parallel nets in Y. By Theorem 3, the function f is uniformly continuous.

As a final example, let $(X, \mathscr{T}_x)$, $(Y, \mathscr{T}_y)$, and $(Z, \mathscr{T}_z)$ be topological vector spaces over K and let $f: X \times Y \longrightarrow Z$ be a bilinear (or a sesquilinear) function. In [3], it was shown that f is continuous if and only if f is C-regular. Using Theorem 3; we can easily show that if f is not identically zero, then f is not uniformly continuous. Fix x_0 in X and y_0 in Y such that $f(x_0, y_0) = z_0 \neq 0$. Since f is bilinear (or sesquilinear), we have $x_0 \neq 0$ and $y_0 \neq 0$. Consider the sequences (nets) (x_n) and (y_n) in $X \times Y$ defined by

$$x_n = \left(n + \frac{1}{n}\right)(x_0, y_0) \text{ and } y_n = n(x_0, y_0).$$

Since $x_n - y_n = \frac{1}{n}(x_0, y_0)$, we have $(x_n - y_n) \longrightarrow (0, 0)$ whence (x_n) and (y_n) are parallel nets in $X \times Y$. On the other hand,

$$f(x_n) - f(y_n) = \left(n + \frac{1}{n}\right)^2 f(x_0, y_0) - n^2 f(x_0, y_0) = \left(2 + \frac{1}{n^2}\right) z_0$$

so $(f(x_n) - f(y_n)) \longrightarrow 2z_0$. Since $2z_0 \neq 0$, the nets $(f(x_n))$ and $(f(y_n))$ in Z are not parallel. By Theorem 3, the function f is not uniformly continuous. A direct verification of this is more tedious. It is rather interesting that a continuous not-identically-zero bilinear (or sesquilinear) function preserves equivalent nets but it does not preserve parallel nets.

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REFERENCES

[1] Hansell, R. W. and Wolk, E. S., *On continuous functions and convergence of nets.* Amer. Math. Monthly 83(1976), 365–366

[2] Kelley, J. L., *General topology,* Van Nostrand, Princeton, N. J., 1955.

[3] Snipes, R. F., *Cauchy-regular functions,* J. Math. Anal. Appl. 79(1981), 18–25.