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A NOTE ON CONTINUATION AND BOUNDEDNESS OF SOLUTIONS OF A NONLINEAR DIFFERENTIAL EQUATION

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A nonlinear second order differential equation is considered. Sufficient conditions for all solutions to be continuable and bounded to the right of an initial value $t_0 \geq 0$ are given:

1. This note is considered with some properties of the solutions of the pesturbed nonlinear second order differential equation.

$$x''+f(x(t))g(x'(t))=h(t,\ x(t),\ x'(t)),\ \ '=\frac{d}{dt} \tag{1}$$

where $f: R \longrightarrow R$, $g: R \longrightarrow R_+$, $h: I \times R^2 \longrightarrow R$ are continuous functions and $R=(-\infty,\ \infty)$, $R_+=(0,\ \infty)$, $I=[0,\ \infty)$.

We shall give sufficient conditions for all solutions of (1) to be continuable to the right of their initial value $t_0 \in I$, and for all solutions $x(t)$ of (1) together with derivative $x'(t)$ to be bounded on I.

DEFINITIONS. (i) By continuable we mean a solution which is defined on a half-line $[t_0,\ \infty)$.

(ii) A solution $x(t)$ of (1) is said to be *oscillatory* if it has no last zero, otherwise it is nonoscillatory.

Let

$$F(x)=\int_0^x f(s)\ ds \geq 0 \text{ for all } x \in R \tag{2}$$

$$G(y)=\int_0^y [s/g(s)]\ ds,\ \ g(0) \neq 0,\ \lim_{|y| \to \infty} G(y)=\infty \tag{3}$$

Our main assumptions are

(i) There exists a continuous function $u: I \longrightarrow R$ such that $h(t,\ x,\ x') \leq u(t)$

(ii) There exists nonnegative constant M such that

$$|y|/g(y) \leq MG\ (y) \text{ for } |y| \geq 1 \tag{4}$$

THEOREM 1. *Under the conditions stated above, the solution $x(t)$ of equation-*

(1) *is continuable to the right of its initial t-value* t_0.

PROOF. Let $x(t)$ be a solution of equation (1) with initial t-value $t_0 \in I$. Suppose, on the contrary, that $x(t)$ can not be continued past the finite point $T > t_0$, $T \in I$. It sufficies to show that $x(t)$ remains bounded as t approaches T from the left.

Let $V(t, x, x') = G(y) + F(x) + C$ where C is a nonnegative constant. Then

$$V'(t\,x,\ x') = \frac{yh}{g(y)} \leq u\,[N + MG(y)]$$

Integrating both sides from t_0 to t and noting that $u(t)$ is bounded on $[t_0, T]$ we have

$$v(t) \leq C_1 + M\int_{t_0}^{t} u(s)\ G(y(s))\ ds$$

Thus

$$G(y(t)) \leq V < C_1 + \int_{t_0}^{t} u(s)\ MG\ (y(s))\ ds.$$

Using Gronwall-Bellman inequality [1] there is a constant C_2 depending on $u(t)$ but not on $G(y)$ such that for all $t\,[t_0, T)$

$$G(y\ (t)) \leq C_2 < \infty.$$

Thus $G(y(t))$ remains bounded as $t \longrightarrow T$ from the left and so $y(t) = x'(t)$ remains bounded as $t \longrightarrow T$ from the left. Consequently $x(t)$ is also bounded as $t \longrightarrow T$. Thus we have a contradiction to our assumption that $x(t)$ is not continued past (finite) point T. This completes the proof.

2. In this section we will prove a boundedness theorem for solution $x(t)$ of equation (1) & their derivatives $x'(t)$ by using a modification of the technique of the previous section. In addition to the given conditions we suppose that

(i) $$\lim_{|x| \to \infty} F(x) = \infty \tag{5}$$

(ii) $$\begin{aligned} & y^2/g(y) \leq MG(y) + N_1 \\ & |y|/g(y) \leq MG(y) + N_2 \quad \text{for all } y \in R \end{aligned} \tag{6}$$

These follow from condition (3).

(iii) There are continuous functions $r_i : I \longrightarrow I$, $i = 1, 2$ such that

$$|h(t, x, y)| \leq r_1(t) + r_2(t)\,|y| \tag{7}$$

for all $(t, x\ y) \in I \times R^2$

THEOREM 2. *Let assumptions* (2), (3), (5), (7) *hold. Let r_1 & r_2 are integrable on I. Then for each solution $x(t)$ of equation* (1) *with initial t-value $t_0 \in I$, ..(t) is bounded on $[t_0, \infty)$. If in addition condition* (5) *holds then $x(t)$ is bounded also on $[t_0, \infty)$.*

PROOF. Let $x(t)$ be a solution of equation (1) with initial t-value t_0, $t_0 \in I$. Multiplying (1) by $x'(t)/g(x(t))$ and integrating on $[t_0, t] \subset [t_0, T)$, we get:

$$G(y(t)) - G(y(t_0)) + F(x(t)) - F(x(t_0))$$
$$\leq \int_{t_0}^{t} h(s,\ x(s),\ y(s))\ y(s)/g(y(s))\ ds \tag{8}$$

Using (6) & (7), inequality(8) can be written in the form

$$G(y(t)) - G(y(t_0) + F(x(t)) - F(x(t_0)) \leq \int_{t_0}^{t} [r_1(s) + r_2(s) y(s)]\ \frac{y(s)}{g(s)}\ ds$$

$$\leq \int_{t_0}^{t} r_1\,(MG(y) + N_1) + r_2\,(Mg(y) + N_2)\ ds$$

$$\leq M \int_{t_0}^{t} [r_1(s) + r_2(s)]\, G(y) ds + \int_{t_0}^{t} (r_1(s) N_1 + r_2(s) N_2)\ ds$$

$$\leq M \int_{t_0}^{t} (r_1(s) + r_2(s)) G(y) ds + m(t) \tag{9}$$

where

$$m(t) = \int_{t_0}^{t} [N_1 r_1(s) + N_2 r_2(s)]\ ds.$$

Furthermore

$$m(t) \leq \int_{t_0}^{\infty} (N_1 r_1(s) + N_2 r_2(s))\ ds = m_0 < \infty \tag{10}$$

Since $F(x) \longrightarrow \infty$ as $|x| \longrightarrow \infty$, $F(x)$ is bounded from below, say $F(x) \geq -K$. Let

$$V(t) = G(y(t)) + F(x(t)) + K \tag{11}$$

Using (11), inequality (9) takes the form

$$V(t) \leq V(t_0) + m(t) + M \int_{t_0}^{t} [r_1(s) + r_2(s)]\, G(y(s))\ ds \tag{12}$$

Hence

$$V(t)\leq V(t_0)+m_0+M\int_{t_0}^{t}[r_1(s)+r_2(s)]V(s)\ ds$$

by using (11) again. By using Gronwall—Bellman inequality there is, then, a constant C, depending on $r_1(t)$ & $r_2(t)$ but not on $V(t)$ such that

$$V(t)\leq(V(t_0)+m_0)\ C,$$

i.e. $(G(y(t))+F(x)\leq[G(y(t_0)+F(x(t_0))+m_0]\ C$. It follow that $F(x(t))$ is bounded for $t\geq t_0$. The conclusion of the theorem follows from (5)

COROLLARY. *If in addition to the hypotheses of Theorem* 2 *and* $G(y)\longrightarrow\infty$ *as* $|y|\longrightarrow\infty$, *then all solutions* $x(t)$ *and the derivatives* $x'(t)$ *are bounded.*

PROOF. From the proof of Theorem 2 we obtain

$$V(t)\leq(V(t_0)+m_0)C<\infty.$$

The boundedness of $y(t)$ then follows from the boundedness of $G(y(t))$ on $[t_0, T)$. An integration shows that $x(t)$ is also bounded on $[t_0, T)$. This completes the proof of the corollary.

THEOREM 3. *If*

$$\begin{aligned}&xf(x)>0,\ \textit{for}\ x\neq0,\ f'(x)\geq0,\\ &h(t,\ x,\ x')\leq u(t),\ \lim_{t\to\infty}u(t)=0,\end{aligned}\tag{14}$$

$$g(y)\geq C>0,\tag{15}$$

and $x(t)$ *is a bounded nonoscillatory solution of* (1), *then* $\liminf_{t\to\infty}|x(t)|=0$

PROOF. It will be convenient to consider the equivalent system

$$\begin{aligned}&x'=y+\int_{t_0}^{t}h(s,\ x(s),\ y(s))\ ds\\ &y'=-f(x)\ g(y).\end{aligned}\tag{16}$$

Let $x(t)$ be a nonoscillatory solution of (1). Without loss of generality, we can assume that $x(t)\neq0$ on $[t_0, \infty)$. Let $x(t)>0$ for $t\geq t_1\geq t_0$. A similar arguments hold if $x(t)<0$ for $t\geq t_1>t_0$. On the contrary, suppose that $\liminf_{t\to\infty} x(t)\neq0$. Then there exist positive numbers m & $t_2\geq t_1$ such that

$$m<x(t)\quad \text{for}\ t\geq t_2\tag{17}$$

This condition together with (14) implies that

$$f(x(t)) \geq A > 0 \quad \text{for } t \geq t_2$$

Thus from (16), by integration, we have

$$y(t) - yt_2 = -\int_{t_2}^{t} f(x(s))\ g(y(s))\ ds$$
$$\leq -AC(t - t_2) \tag{18}$$

Letting $t \longrightarrow \infty$ in (18) we obtain

$$\lim_{t \to \infty} y(t) = -\infty$$

Thus, for $t \geq t_3 \geq t_2$ for some t_3 sufficiently large

$$x'(t) < 0 \quad \text{for } t \geq_{t3}$$

Then from (16), by integration, it follows that

$$x(t) \longrightarrow -\infty \quad \text{as } t \longrightarrow \infty.$$

This, however, is a contradiction and hence

$$\liminf_{t \to \infty} x(t) = 0.$$

This completes the proof of the theorem.

REMARKS (1) It should be noted that x' is not necessarily bounded on $[t_0, \infty)$; this follows from the fact that the equation

$$x'' + e^{2t}\ x = x'$$

has the bounded solution $x(t) = \sin(e^t)$ with an unbounded derivative.

(2) No restriction condition on the forcing term $h(t, x, x')$ to be small is required.

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REFERENCES

[1] W. Schmaedeke & G. Sell. *The Gronwall inequality for modified stieltjes integrals*, Proc. Amer. Math. Soc. 19(1968), 1217–1222.

[2] H. El-Owaidy, *On oscillations of second order differential equalion*, Kyungpook Math. J. V. 21 (1981) 117–121.