Kyungpook Math. J.
Volume 23, Number 1
June, 1983

ON THE SPECIAL CLASSES OF p-VALENT FUNCTIONS

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1. Introduction

Let A denote the family of functions

$$f(z)=z^p+\sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p\in N)$$

analytic in the unit disk $U=\{|z|<1\}$. Let $f*g(z)$ denote the Hadamard product of two functions

$$f(z)=z^p+\sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p\in N)$$

and

$$g(z)=z^p+\sum_{n=1}^{\infty} b_{p+n} z^{p+n} \qquad (p\in N)$$

in the family A, that is,

$$f*g(z)=z^p+\sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}.$$

Furthermore, let

$$D^{p+\alpha-1}f(z)=\frac{z^p}{(1-z)^{p+\alpha}}*f(z)$$

for $0<\alpha<p$ and $p\in N$.

R. M. Goel and N. S. Sohi [1] studied the classes $T_{n+p-1}(\alpha)$ of functions

$$f(z)=z^p+\sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p\in N)$$

analytic in the unit disk U and satisfying

$$\mathrm{Re}\frac{\{D^{n+p-1}f(z)\}'}{pz^{p-1}}>\alpha$$

for $0\leqq\alpha<1$ and $z\in U$.

In this paper, let $T_{p+\alpha-1}(\beta)$ and $T_{p-\alpha-1}(\beta)$ denote the classes of functions

$$f(z)=z^p+\sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p\in N)$$

in the family A satisfying the conditions

$$\mathrm{Re}\frac{\{D^{p+\alpha-1}f(z)\}'}{pz^{p-1}}>\beta$$

for $0<\alpha<1$, $0\leqq\beta<1$ and $z\in U$ and

$$\mathrm{Re}\frac{\{D^{p-\alpha-1}f(z)\}'}{pz^{p-1}}>\beta$$

for $0<\alpha<p$, $0\leqq\beta<1$ and $z\in U$, respectively.

2. The fractional calculus

There are many definitions of the fractional calculus. In 1978, S. Owa [4] gave the following definitions for the fractional calculus.

DEFINITION 1. The fractional integral of order α is defined by

$$D_z^{-\alpha}f(z)=\frac{1}{\Gamma(\alpha)}\int_0^z\frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\alpha}},$$

where $\alpha>0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring log $(z-\zeta)$ to be real when $(z-\zeta)>0$. Moreover,

$$f(z)=\lim_{\alpha\to 0}D_z^{-\alpha}f(z).$$

DEFINITION 2. The fractional derivative of order α is defined by

$$D_z^{\alpha}f(z)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z\frac{f(\zeta)d\zeta}{(z-\zeta)^{\alpha}},$$

where $0<\alpha<1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$. Moreover,

$$f(z)=\lim_{\alpha\to 0}D_z^{\alpha}f(z).$$

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\alpha)$ is defined by

$$D_z^{n+\alpha}f(z)=\frac{d^n}{dz^n}D_z^{\alpha}f(z),$$

where $0<\alpha<1$ and $n\in N\cup\{0\}$.

For other definitions of the fractional calculus, see K. Nishimoto [2], T. J. Osler [3], B. Ross [6] and M. Saigo [7].

S. Owa [5] showed the following results for the fractional calculus.

LEMMA 1. *Let the function*

$$f(z)=z^{p}+\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (p\in N)$$

be in the family A. Then we have

$$D^{p+\alpha-1}f(z)=\frac{z^{p}}{\Gamma(p+\alpha)}D_{z}^{p+\alpha-1}\{z^{\alpha-1}f(z)\}$$

and

$$D^{p-1}f(z)=\lim_{\alpha\to 0}D^{p+\alpha-1}f(z)$$

for $0<\alpha<1$ *and* $z\in U$.

LEMMA 2. *Let the function*

$$f(z)=z^{p}+\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (p\in N)$$

be in the family A. Then we have

$$D^{p-\alpha-1}f(z)=\frac{z^{p}}{\Gamma(p-\alpha)}D_{z}^{p-\alpha-1}\{z^{-\alpha-1}f(z)\}$$

and

$$D^{p-1}f(z)=\lim_{\alpha\to 0}D^{p-\alpha-1}f(z)$$

for $0<\alpha<p$ *and* $z\in U$.

3. Properties of the classes $T_{p+\alpha-1}(\beta)$ and $T_{p-\alpha-1}(\beta)$

THEOREM 1. *Let* $0<\alpha<1$ *and* $0\leqq\beta_1\leqq\beta_2<1$. *Then we have*

$$T_{p+\alpha}(\beta_2)\subset T_{p+\alpha-1}(\beta_1).$$

PROOF. By using the same technique as in the proof of

$$T_{n+p}(\alpha)\subset T_{n+p-1}(\alpha)$$

in [1], we have

$$T_{p+\alpha}(\beta_2)\subset T_{p+\alpha-1}(\beta_2).$$

Furthermore, by the definition of the class $T_{p+\alpha-1}(\beta)$,

$$T_{p+\alpha}(\beta_2)\subset T_{p+\alpha-1}(\beta_2)\subset T_{p+\alpha-1}(\beta_1)$$

for $0\leqq\beta_1\leqq\beta_2<1$.

THEOREM 2. *Let* $0<\alpha<1$ *and* $0\leqq\beta_1\leqq\beta_2<1$. *Then we have*

$$T_{p-\alpha}(\beta_2)\subset T_{p-\alpha-1}(\beta_1).$$

The proof of Theorem 2 is obtained by using the same technique as in the proof of Theorem 1.

4. Applications for the fractional calculus

In 1973, D. B. Shaffer [8] showed the following lemma.

LEMMA 3. *Let the function*

$$h(z)=1+a_p z^p+a_{p+1} z^{p+1}+\cdots \qquad (p\in N)$$

be analytic in the unit disk U *and* $\mathrm{Re}\{h(z)\}>k$ $(0\leqq k<1)$.
Then

$$|h'(z)|\leqq\frac{2(1-k)}{(1-|z|)^2}.$$

THEOREM 3. *Let the function*

$$f(z)=z^p+\sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p\in N)$$

be in the class $T_{p+\alpha-1}(\beta)$ *for* $0<\alpha<1$ *and* $0\leqq\beta<1$ *and satisfying*

$$\sum_{n=1}^{\infty}\frac{(p+n)\Gamma(p+n+\alpha)}{n!\,\Gamma(p+\alpha)}|a_{p+n}|\leqq M,$$

where M *is a constant. Then we have*

$$|D_z^{p+\alpha-1}\{z^{\alpha-1}f(z)\}|\geqq\Gamma(p+\alpha)\left(1-\frac{M}{p+1}|z|\right)$$

and

$$|D_z^{p+\alpha-1}\{z^{\alpha-1}f(z)\}|\leqq\Gamma(p+\alpha)\left(1+\frac{M}{p+1}|z|\right)$$

for $z\in U$ *and*

$$|D_z^{p+\alpha}\{z^{\alpha-1}f(z)\}\leqq\Gamma(p+\alpha)\left(\frac{2p}{|z|}+\frac{2p+1}{p+1}M\right)$$

and

$$|D_z^{p+\alpha+1}\{z^{\alpha-1}f(z)\}|\leqq\Gamma(p+\alpha)\left\{\frac{2p(1-\beta)}{(1-|z|)^2|z|}+\frac{2p(3p-1)}{|z|^2}+\frac{(2p+1)(3p-1)}{p+1}M\right\}$$

for $z\in U-\{0\}$.

PROOF. By using Lemma 1, we have

$$D^{p+\alpha-1}f(z)=\frac{z^p}{\Gamma(p+\alpha)}\,D_z^{p+\alpha-1}\,\{z^{\alpha-1}f(z)\}$$

$$=z^p+\sum_{n=1}^{\infty}\frac{\Gamma(p+n+\alpha)}{n!\,\Gamma(p+\alpha)}a_{p+n}\,z^{p+n}.$$

Hence,

$$|D_z^{p+\alpha-1}\,\{z^{\alpha-1}f(z)\}|\leqq\Gamma(p+\alpha)+|z|\sum_{n=1}^{\infty}\frac{\Gamma(p+n+\alpha)}{n\,!}|a_{p+n}|$$

$$\leqq\Gamma(p+\alpha)+\frac{\Gamma(p+\alpha)M}{p+1}|z|$$

and

$$|D_z^{p+\alpha-1}\,\{z^{\alpha-1}f(z)\}|\geqq\Gamma(p+\alpha)-\frac{\Gamma(p+\alpha)M}{p+1}|z|.$$

In the second place, by a simple calculation, we have

$$\{D^{p+\alpha-1}f(z)\}'=pz^{p-1}+\sum_{n=1}^{\infty}\frac{(p+n)\Gamma(p+n+\alpha)}{n\,!\,\Gamma(p+\alpha)}\,a_{p+n}\,z^{p+n-1}.$$

Therefore, by using the condition of the theorem,

$$|\{D^{p+\alpha-1}f(z)\}'|\leqq p|z|^{p-1}+M|z|^p.$$

On the other hand, we have

$$\{D^{p+\alpha-1}f(z)\}'=\frac{pz^{p-1}}{\Gamma(p+\alpha)}\,D_z^{p+\alpha-1}\,\{z^{\alpha-1}f(z)\}+\frac{z^p}{\Gamma(p+\alpha)}\,D_z^{p+\alpha}\{z^{\alpha-1}f(z)\}$$

with the aid of Lemma 1. Consequently, we have the third estimate with the second estimate.

Finally, since the function $f(z)$ is in the class $T_{p+\alpha-1}(\beta)$,

$$\mathrm{Re}\frac{\{D^{p+\alpha-1}f(z)\}'}{pz^{p-1}}>\beta$$

and

$$\frac{\{D^{p+\alpha-1}f(z)\}'}{pz^{p-1}}=1+\sum_{n=1}^{\infty}\frac{(p+n)\Gamma(p+n+\alpha)}{pn\,!\,\Gamma(p+\alpha)}a_{p+n}\,z^n$$

is analytic in the unit disk U, that is, $\{D^{p+\alpha-1}f(z)\}'/pz^{p-1}$ meets the conditions Lemma 3. Hence,

$$|\{D^{p+\alpha-1}f(z)\}''|\leqq\frac{2p(1-\beta)|z|^{p-1}}{(1-|z|)^2}+p(p-1)|z|^{p-2}+(p-1)M|z|^{p-1}$$

by Lemma 3. Furthermore, by using the second estimate and the third estimate, we have the final estimate.

THEOREM 4. *Let the function*

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (p\in N)$$

be in the class $T_{p-\alpha-1}(\beta)$ *for* $0<\alpha<1$ *and* $0\leqq\beta<1$ *and satisfying*

$$\sum_{n=1}^{\infty}\frac{(p+n)\Gamma(p+n-\alpha)}{n!\,\Gamma(p-\alpha)}|a_{p+n}|\leqq M,$$

where M *is a constant. Then we have*

$$|D_z^{p-\alpha-1}\{z^{-\alpha-1}f(z)\}|\geqq\Gamma(p-\alpha)\Big(1-\frac{M}{p+1}|z|\Big)$$

and

$$|D_z^{p-\alpha-1}\{z^{-\alpha-1}f(z)\}|\leqq\Gamma(p-\alpha)\Big(1+\frac{M}{p+1}|z|\Big)$$

for $z\in U$ *and*

$$|D_z^{p-\alpha}\{z^{-\alpha-1}f(z)\}|\leqq\Gamma(p-\alpha)\Big(\frac{2p}{|z|}+\frac{2p+1}{p+1}M\Big)$$

and

$$|D_z^{p-\alpha+1}\{z^{-\alpha-1}f(z)\}|\leqq\Gamma(p-\alpha)\Big\{\frac{2p(1-\beta)}{(1-|z|)^2|z|}+\frac{2p(3p-1)}{|z|^2}+\frac{(2p+1)(3p-1)}{p+1}M\Big\}$$

for $z\in U-\{0\}$.

The proof of Theorem 4 is given in much the same way as the proof of Theorem 3 with the aids of Lemma 2 and Lemma 3.

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