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EXTENDING CERTAIN SEMIRING HOMOMORPHISMS TO RING HOMOMORPHISMS

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1. Introduction

One of the more interesting aspects of any algebraic structure is the study of homomorphisms of that structure. It is usually interesting to see what properties of a structure are preserved under homomorphisms. In this paper we will be cencerned with extending certain semiring homomorphisms to ring homomorphisms and determining what properties of the semiring homomorphism are preserved under the extension. The results that we obtain will be applied to a proof of a universal mapping property of halfrings. In order to extend a semiring homomorphism to a ring homomorphism we must first embed the semiring in a ring.

2. Fundamentals

A semiring is a set S together with two binary operations called addition $(+)$ and multiplication $\cdot$ such that $(S, +)$ is an abelian semigroup with a zero, $(S, \cdot)$ is a semigroup, and multiplication distributes over addition from both the left and the right. In order that a semiring S be embedded in a ring it is necessary and sufficient that S be cancellative. We call a cancellative semiring a halfring. To embed a semiring H in a ring we proceed as follows. Let $H^*=\{(h, k)|h, k\in H\}$ and in H^* define $(h,k)=(h', k')$ if and only if $h+k'=h'+k$. This gives an equivalence relation on H^*. Let $\overline{H}$ be the set of equivalence classes in H^*. In $\overline{H}$ define

$$(h, k)+(h', k')=(h+h', k+k') \text{ and}$$
$$(h, k)(h', k')=(hh'+kk', hk'+kk'),$$

then $\overline{H}$ is a ring with respect to these operations. The map $\phi: H\longrightarrow\overline{H}$ given by $\phi(h)=(h, 0)$ is a well defined injection and it follows that H is embedded in $\overline{H}$. We identify the ordered pair (h, k) with $h-k$. Then $\overline{H}=\{h-k|h, k\in H\}$ is called *the ring of difference* of H and is the smallest ring containing H. Since $\overline{H}$ is the smallest ring containing H, we will call $\overline{H}$ the closure of H.

A nonempty subset I of a semiring S is called an *ideal in* S if I is a subs emiring of S and $SI\subset I$ and $IS\subset I$. An ideal I in a semiring S is called a *k-ideal* if $a\in I$, $b\in S$ and $a+b\in I$ imply $b\in I$. These ideals are also called *subtractive.* Now if A is an ideal in a halfring H, then it is easy to see that $\overline{A}=\{a_1-a_2|a_i\in A\}$ is an ideal in $\overline{H}$. Conversely, if B is an ideal in $\overline{H}$, then $B\cap H$ is an ideal in H. It is not generally true that if A is an ideal in H, then $A=\overline{A}\cap H$. This can happen if and only if A is a *k-ideal.*

3. Halfrings and homomorphisms

Let H and K be halfrings, $\overline{H}$ and $\overline{K}$ their closures, and $f: H\longrightarrow K$ a halfring homomorphism. Define $\bar{f}: \overline{H}\longrightarrow\overline{K}$ by $\bar{h}\bar{f}=(h_1-h_2)\bar{f}=h_1f-h_2f$, where $\bar{h}=h_1-h_2$.

THEOREM 3.1. *If $f: H\longrightarrow K$ is a halfring homomorphism, then $\bar{f}: \overline{H}\longrightarrow\overline{K}$ is a ring homomorphism such that $h\bar{f}=hf$ for all $h\in H$.*

PROOF. First we show that $\bar{f}$ is well-defined. Suppose $x=x_1-x_2$, $y=y_1-y_2\in\overline{H}$ and $x=y$. Then $x_1+y_2=x_2+y_1$ and it follows that $x_1f+y_2f=(x_1+y_2)f=(x_2+y_1)f=x_2f+y_1f$. Consequently,

$$x\bar{f}=(x_1-x_2)\bar{f}=x_1f-x_2f=y_1f-y_2f=(y_1-y_2)\bar{f}=y\bar{f}$$

and it follows that $\bar{f}$ is well defined. Now

$$\begin{aligned}(x+y)\bar{f}&=[(x_1-x_2)+(y_1-y_2)]\bar{f}\\&=[(x_1+y_1)-(x_2+y_2)]\bar{f}\\&=(x_1+y_1)f-(x_2+y_2)f\\&=x_1f+y_1f-(x_2f+y_2f)\\&=(x_1f-x_2f)+(y_1f-y_2f)\\&=(x_1-x_2)\bar{f}+(y_1-y_2)\bar{f}\\&=x\bar{f}+y\bar{f}\end{aligned}$$

and

$$\begin{aligned}(xy)\bar{f}&=[(x_1-x_2)(y_1-y_2)]\bar{f}\\&=[(x_1y_1+x_2y_2)-(x_1y_2+x_2y_1)]\bar{f}\\&=(x_1y_1+x_2y_2)f-(x_1y_2+x_2y_1)f\\&=(x_1f)(y_1f)+(x_2f)(y_2f)-(x_1f)(y_2f)-(x_2f)(y_1f)\\&=(x_1f-x_2f)(y_1f-y_2f)\\&=(x_1-x_2)\bar{f}(y_1-y_2)\bar{f}=(x\bar{f})(y\bar{f}).\end{aligned}$$

Therefore $\bar{f}$ is a ring homomorphism. Now if $h \in H$, then $h=h-0$ and $h\bar{f}=(h-0)\bar{f}=hf-0f=hf$.

DEFINITION 3.2. If $f: H \longrightarrow K$ is a halfring homomorphism, then the map $\bar{f}: \overline{H} \longrightarrow \overline{K}$ given in Theorem 3.1 is called the *extension* of f to $\overline{H}$.

Now suppose $g: \overline{H} \longrightarrow \overline{K}$ is a homomorphism of rings. Then g_H, the restriction of g to H is a homomorphism. If $x=x_1-x_2 \in \overline{H}$, then

$$xg=x_1g-x_2g=x_1g_H-x_2g_H=(x_1-x_2)\bar{g}_H=x\bar{g}_H$$

and it follows that $\bar{g}_H=g$. Thus each halfring homomorphism $f: H \longrightarrow K$ induces a ring homomorphism $\bar{f}: \overline{H} \longrightarrow \overline{K}$ and conversely each ring homomorphism $g: \overline{H} \longrightarrow \overline{K}$ gives a halfring homomorphism $g_H: H \longrightarrow K$ such that $\bar{g}_H=g$. If H and K are halfrings and $f,\ g \in \mathrm{Hom}(H, K)$, then $\bar{f},\ \bar{g} \in \mathrm{Hom}(\overline{H}, \overline{K})$. It is clear that $\mathrm{Hom}(H, K)$ is semigroup under addition defined by
$a(f+g)=af+ag$. Likewise $\mathrm{Hom}(\overline{H}, \overline{K})$ is a group. Let $x=x_1-x_2 \in \overline{H}$. Then

$$\begin{aligned} x(\bar{f}+\bar{g}) &= x\bar{f}+x\bar{g} \\ &= (x_1f-x_2f)+(x_1g-x_2g) \\ &= (x_1f+x_1g)-(x_2f+x_2g) \\ &= x_1(f+g)-x_2(f+g) \\ &= (x_1-x_2)(\overline{f+g})=x(\overline{f+g}). \end{aligned}$$

Consequently, $\bar{f}+\bar{g}=\overline{f+g}$. Thus the map $\Psi: \mathrm{Hom}(H, K) \longrightarrow \mathrm{Hom}(\overline{H}, \overline{K})$ given by $f\Psi=\bar{f}$ is a homomorphism. This proves the following theorem.

THEOREM 3.3. *If H and K are halfrings with closures $\overline{H}$ and $\overline{K}$, then the map $\Psi: \mathrm{Hom}(H, K) \longrightarrow \mathrm{Hom}(\overline{H}, \overline{K})$ given by $f\Psi=\bar{f}$ is an isomorphism.*

We want to show now that f is an isomorphism if and only if $\bar{f}$ is an isomorphism. If $f: H \longrightarrow K$ and $\bar{f}$ is injective it is clear that f is injective since $hf=h\bar{f}$ for all $h \in H$. On the other hand, suppose f is injective and $\bar{h}\bar{f}=(h_1-h_2)\bar{f}=0$. Then $\bar{h}\bar{f}=h_1f-h_2f=0$ and it follows that $h_1f=h_2f$. But f is injective so that $h_1=h_2$. Consequently, $\bar{h}=h_1-h_2=0$ and it follows that $\bar{f}$ is injective. Now if $\bar{f}$ is surjective then it is clear that f is surjective since $K \subset \overline{K}$. On the other hand, suppose f is surjective and $y \in \overline{K}$ with $y=y_1-y_2$. Then there exists $x_1,\ x_2 \in H$ such that $x_1f=y_1$ and $x_2f=y_2$. Consequently,

$$y=y_1-y_2=x_1f-x_2f=(x_1-x_2)\bar{f}=x\bar{f}$$

and it follows that $\bar{f}$ is surjective. This proves the following theorem.

THEOREM 3.4. *If* $f: H \longrightarrow K$ *is a halfring homomorphism with extension* $\bar{f}$, *then* f *is an isomorphism if and only if* $\bar{f}$ *is an isomorphism.*

We want to consider now compositions of homomorphisms and their extensions.

THEOREM 3.5. *Let* $f: H \longrightarrow K$ *and* $g: K \longrightarrow L$ *be halfring homomorphisms with extensions* $\bar{f}$ *and* $\bar{g}$. *Then*

1. $\overline{fg} = \bar{f}\bar{g}$
2. $\bar{1}_H = 1_{\bar{H}}$
3. $(\bar{f})^{-1} = (\overline{f^{-1}})$ *if* f^{-1} *exists.*

PROOF. (1) Let $\bar{h} = h_1 - h_2 \in \bar{H}$. Then $\bar{h}(\bar{f}\bar{g}) = (\bar{h}\bar{f})\bar{g} = [(h_1 - h_2)\bar{f}]\bar{g} = (h_1 f - h_2 f)\bar{g} = (h_1 f)g - (h_2 f)g = h_1(fg) - h_2(fg) = (h_1 - h_2)(\overline{fg}) = \bar{h}(\overline{fg})$. Consequently, we have $\bar{f}\bar{g} = \overline{fg}$.

(2) Now $\bar{h}\bar{1}_H = (h_1 - h_2)\bar{1}_H = h_1 1_H - h_2 1_H = h_1 - h_2 = \bar{h} \equiv \bar{h} 1_{\bar{H}}$ and it follows that $\bar{1}_H = 1_{\bar{H}}$.

(3) From (1) and (2) we have $\bar{f}(\overline{f^{-1}}) = (\overline{ff^{-1}}) = \bar{1}_H = 1_{\bar{H}}$ and $(\overline{f^{-1}})\bar{f} = (\overline{f^{-1}f}) = \bar{1}_K = 1_{\bar{K}}$ and it follows that $(\overline{f^{-1}}) = (\bar{f})^{-1}$.

We now give an application of Theorem 3.5. Recall that an exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is said to *split* if there exist $\gamma: B \longrightarrow A$ such that $\alpha\gamma = 1_A$.

THEOREM 3.6. *Suppose*

$$A: 0 \longrightarrow H \xrightarrow{\alpha} K \xrightarrow{\beta} L \longrightarrow 0 \text{ and } B: 0 \longrightarrow \bar{H} \xrightarrow{\bar{\alpha}} \bar{K} \xrightarrow{\bar{\beta}} \bar{L} \longrightarrow 0$$

are exact sequences. Then the sequence A *splits if and only if the sequence* B *splits.*

PROOF. If B splits, then it is obvious that A splits since $\alpha = \bar{\alpha}_H$. Now suppose A splits and $\gamma: K \longrightarrow H$ such that $\alpha\gamma = 1_H$. Then $\bar{\gamma}: \bar{K} \longrightarrow \bar{H}$ and by Theorem 3.5, $\bar{\alpha}\bar{\gamma} = \overline{\alpha\gamma} = \bar{1}_H = 1_{\bar{H}}$ and it follows that B splits.

4. A uuiversal property for halfrings

A nonempty subset S of a ring R is called *multiplicative* provided $a, b \in S$ implies $ab \in S$. If S is a multiplicative subset of a commutative ring R, then the relation defined on the set $R\times S$ by $(r, s)\sim(r', s')$ if and only if $t(rs'-r's)=0$ for some $t\in S$ is an equivalence relation. If we denote the equivalence class (r, s) by $\frac{r}{s}$, then the set of equivalence classes of $R\times S$, denoted R_S, is a ring under the operations of addition and multiplication defined by

$$\frac{r}{s}+\frac{r'}{s'}=\frac{rs'+r's}{ss'} \text{ and } \left(\frac{r}{s}\right)\left(\frac{r'}{s'}\right)=\frac{rr'}{ss'}$$

respectively. Also, the mapping $\phi_s : R\longrightarrow R_S$ given by $r\phi_s=\frac{rs}{s}$ (for any $s\in S$) is a well defined homomorphism of rings such that $s\phi_s$ is a unit in R_S for every $s\in S$. If H is a halfring and S is a multiplicative subset of H, then it is rather straightforward to show that H_S is a halfring. If P is a prime ideal in H, then $\overline{P}$ is a prime ideal in $\overline{H}$. Consequently, P and $\overline{P}$ are multiplicative subsets of H and $\overline{H}$ respectively. If we let $S=P-H$ and $\overline{S}=\overline{P}-\overline{H}$, then from the definition of prime ideal it follows that S and $\overline{S}$ are multiplicative subsets of H and $\overline{H}$ respectively. The universal mapping property for rings states that if S is a multiplicative subset of a commutative ring R, T is a commutative ring with identity, and $g : R\longrightarrow T$ is a homomorphism such that sg is a unit in T for all $s\in S$, then there exists a unique homomorphism $g^* : R_S\longrightarrow T$ such that $\phi_s g^*=g$. Our aim is to extend this property to halfrings using the results in section 3.

THEOREM 4.1. *Let P be a prime ideal in a commutative halfring H, $S=P-H$, and K a commutative halfring with an identity. If $f : H\longrightarrow K$ is a halfring homomorphism such that sf is a unit in K for all $s\in S$, then there exists a unique halfring homomorphis$_m$ $\Psi : H_S\longrightarrow K$ such that $\phi_s\Psi=f$. The halfring H_S is completely determined by this property.*

PROOF. Let $\bar{f} : \overline{H}\longrightarrow\overline{K}$ be the extension of f. Now $\overline{P}$ is a prime ideal in $\overline{H}$. Let $\overline{S}=\overline{P}-\overline{H}$. Then $\overline{H}_{\overline{S}}$ is a commutative ring with identity. Let $\bar{\phi}_s : \overline{H}\longrightarrow\overline{H}_{\overline{S}}$ be the extension of $\phi_s : H\longrightarrow H_S$. Now $s\bar{f}=sf$ is a unit in $\overline{K}$ for all $s\in S$. Thus by the universal mapping property for rings, there exists a unique homomorphism $\overline{\Psi} : \overline{H}_{\overline{S}}\longrightarrow\overline{K}$ such that $\bar{\phi}_S\overline{\Psi}=\bar{f}$. By Theorem 3.3, $\mathrm{Hom}(H_S, K)\simeq\mathrm{Hom}(\overline{H}_{\overline{S}}, \overline{K})$. Hence $\Psi : H_S\longrightarrow K$, where $\Psi=\overline{\Psi}_{H_s}$ is the unique homomorphism such that ${}_{\phi S}\Psi=f$. The map $\overline{\Psi}$ is given by $\left(\frac{\bar{r}}{\bar{s}}\right)\overline{\Psi}=\bar{f}(\bar{r})\bar{f}(\bar{s})^{-1}$ and it follows that Ψ is given by $\left(\frac{r}{s}\right)\Psi=f(r)f(s)^{-1}$.

Thus we can extend the universal mapping property of ring to halfrings.

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