HONAM MATHEMATICAL JOURNAL
Volume 5, Number 1, September, 1983

A Note on the Gauss Map of a Complete Minimal Surface in R^3

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0. INTRODUCTION

An immersed submanifold M into N is called minimal if its mean curvature vector vanishes at every point. Using the method of calculus of variations, we can show that for an immersion $f:M \longrightarrow N$ of a compact oriented manifold M with boundary into N, the immersion f is a critical point for the volume function $V(g)$, among all immersions $g: M \longrightarrow N$ with $g=f$ on the boundary of M, if and only if M is a minimal (immersed) submanifold ([3] or [7]). In particular, if f is volume-minimizing among all such immersions, then f is minimal. Thus the study of the minimal submanifold may be regarded as a generalization of the study of geodesics, because of the fact that a piecewise smooth curve C is a geodesic if and only if C is a critical point for the length function or equivalently for the energy function [(4] or [6]). As well as the volume-minimizing properties, minimal submanifolds have much interesting properties. Among these properties we consider mainly the properties which the Gauss map of a complete minimal submanifold has. A wellknown theorem of Osserman states that the image of the Gauss map of a complete nonflat regular minimal surface in R^3 is dense in S^2 ([5], p. 68). It can be proved that every flat minimal surface in R^n is a plane ([3], p. 116] and that no minimal surface without boundary in R^n can be compact ([3], p. 14). So we consider only connected nonflat (noncompact) minimal surfaces in R^3.

In this note we represent some typical examples and interesting properties of minimal surfaces in R^n, and we improve the Osserman's Theorem by showing that the Gauss map of a complete regular minimal surface in R^3 omits at most six points, following the Xavier's paper ([10]) with correcting some errata. For any set of k points in S^2 where $k \leq 4$, there are examples whose Gauss map omits exactly the set in S^2 (Theorem 3). But no examples have been known where the omitted sets have 5 (or 6) points. Throughout this note, all surfaces will be connected and oriented submanifolds of R^3 with the induced metric.

I. PRELIMINARIES AND EXAMPLES

We state a theorem which plays a major role in the theory of minimal surfaces in R^3 and which allows us to construct a great variety of minimal surfaces in R^3.

THEOREM 1. (Weierstrass Representation Theorem of Minimal Surfaces) Let D be a domain in the complex plane, g an arbitrary meromorphic function in D and f an analytic function in D having the property that at each point where g has a pole of order m, f has a zero of order at least 2m. set $\phi_1=f(1-g^2)/2$, $\phi_2=if(1+g^2)/2$, $\phi_3=fg$.
Suppose that the analytic functions ϕ_k have no real periods on D, *i.e.* the real part of the integral of ϕ_k on a curve in D depends only on the end points. (In particular, if D is simply connected, every analytic function in D has no real periods.) Then the function $x=(x_1, x_2, x_3): D \longrightarrow R^3$ will define a minimal surface M in R^3 whose metric is given by $ds^2=\lambda^2|dz|^2$, where $\lambda=|f|(1+|g|^2)/2$ and $x_k(z)=Re(\int^z \phi_k(w)\, dw) \cdots (*)$ And M is regular if and only if f satisfies the further properties that it vanishes only at the poles of g, and the order of its zero at such a point is exactly twice of the order of the pole of g.

Proof. The proofs can be found in [3] and [5].

For simply connected minimal surfaces, we can prove the following, using the Koebe Uniformization Theorem ([5]).

THEOREM 2. Every simply connected minimal surface M in R^3 can be represented in the form (*), where the domain D is either the unit disk or the entire plane.

THEOREM 3. ([8]) Let M be an immersed surface in R^3 with the Gauss map n. If M is minimal, then n is conformal (angle-preserving) at all points where the curvature $k\neq 0$. Conversely, if n is conformal, and M is connected, then either M is a minimal surface, with $k<0$, or M is part of a sphere.

Let $\pi: S^2-\{(0,0,1)\} \longrightarrow R^2$ be the stereographic projection and let M be the surface in the theorem 1. Then it can be shown that $g=\pi\circ n\circ x$, where n is the Gauss map of M. ([3], p. 113 or [5], p. 66). Hence the poles of g

occur exactly at those points $p \in M$ where $n(p) = (0, 0, 1)$. Thus, if n omits at least one point of S^2, we may assume, by making a rotation of coordinates, that g has no poles on D (*i. e.* g is analytic in D).

Now we give some fundamental examples of complete minimal surface in R^3. The representation of these examples in the form in Theorem 1 can be found in [3] and [8].

(1) Plane : $ax + by + cz = d$, where $(a, b, c) \neq (0, 0, 0)$.

The Gauss map of this surface is constant. And this is the unique complete flat minimal surface in R^3.

(2) Catenoid : $(\cosh \frac{x}{c})^2 = (y/c)^2 + (z/c)^2$, $(c \neq 0)$ constant.

The Gauss map omits 2 points $(\pm 1, 0, 0)$. This is the only complete minimal surface which is also a surface of revolution. ([3], [7]).

(3) Helicoid : $\frac{y}{x} = \tan \frac{z}{c}$, $c \neq 0$ constant.

The Gauss map omits 2 points $(0, 0, \pm 1)$. This is the only complete ruled minimal surface. ([3], [7]).

(4) Scherk's surface : $e^z \cos x = \cos y$.

The Gauss map omits four points $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$. ([3], [7]).

(5) Enneper's surface : $x = Re(w - w^3/3)$, $y = Re(i(w + w^3/3))$, $z = Re(w^2)$, where w ranges over the complex plane. The Gauss map omits one points (0, 0, 1).

(6) Schwarz surface ([3], p. 104)

(7) Gyroid

This is an infinitely connected periodic minimal surface containing no straight lines, recently discovered and christened by A. H. Schoen. (See the picture in [5]).

We have given examples of minimal surfaces whose Gauss map omits 1, 2 and 4 points. But, alternatively, using Theorem 1, we can get the following. The proof can be found in [5].

THEOREM 4. Let E be an arbitrary set of k points on S^2, where $k \leq 4$. Then there exists a complete regular minimal surface in R^3 whose image under the Gauss map omits precisely the set E.

2. DEFINITIONS AND LEMMAS

DEFINITION 1. A function meromorphic in the unit disk is called normal if the family of functions $\{f(e^{it}\frac{z+z_0}{1+\bar{z}_0 z}) \mid t \in R, \ |z_0| < 1\}$ is normal in Montel's sense, *i. e.* any sequence in the family contains a subsequence converging uniformly on compact subsets of the unit disk.

LEMMA 1. A function $f(z)$ meromorphic in the unit disk is normal if and only if $\frac{(1-|z|^2)|f'(z)|}{1+|f(z)|^2} \leqslant C \quad (|z|<1)$, where C is a constant.

Proof. See the Theorem 6.5 in [1].

Lemma 2. Let f be a holomorphic function in the unit disk D and let $f \neq 0$, $a(\neq 0)$. Let $\alpha = 1-1/k$, $k \in Z^+$. Then we have $\frac{|f'|}{|f|^{\alpha}+|f|^{2-\alpha}} \in L^p(D)$ for every p with $0<p<1$.

Proof. Since $f^{1/k}$ omits two values, it is normal (see [1], p. 169). Using the lemma 1 for $f^{1/k}$, we can show that

$$\frac{|f'|}{k|f|^{1-1/k}(1+|f|^{2/k})} \leqslant \frac{C}{1-|z|^2} \quad \text{so that} \quad \frac{|f'|}{|f|^{\alpha}+|f|^{2-\alpha}} \leqslant \frac{kC}{1-|z|^2}$$

Hence the fact that $(1-|z|^2)^{-1} \in L^p(D)$ for $0<p<1$ completes the proof.

DEFINITION 2. Let M be a connected Riemannian n-dimensional manifold. The Laplace-Beltrami operator on M is a map $\Delta : C^\infty(M) \to C^\infty(M)$ defined by the formula (1), or equivalently by (2)

(1) $\Delta f = -{*d}({*df})$, where $*$ is the Hodge star operator.

(2) $\Delta f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial x^j})$, where $(x^1, \cdots, x^n)$ are local coordinates, the metric $ds^2 = \sum g_{ij}\, dx^i\, dx^j$, the matrix $(g^{ij}) = (g_{kl})^{-1}$ and $g = det\,(g_{ij})$.

LEMMA 3. Let M be a complete Riemannian manifold and u a nonconstant and nonnegative function satisfying $\Delta \log u = 0$ almost everywhere.
Then $\int_M u^p = \infty$ for $p>0$.

Proof. See the Theorem 1 in [9].

3. MAIN THEOREM

THEOREM 5 The Gauss map of a complete nonflat minimal surface in R^3 omits at most 6 points of S^2.

Proof. Suppose that M is a complete nonflat minimal surface in R^3 whose Gauss map omits at least 7 points.
By passing to the universal covering surface we may assume that M is simply

connected. By Theorem 2 M can be represented in the form (*) where the domain D is either the unit disk or the entire plane.

Since the Gauss map omits at least one point and M is not flat, we may assume that g is a nonconstant holomorphic function. Since M is regular, the holomorphic function $f \neq 0$. If D is the entire plane, then by the Picard Theorem g assumes all complex values with at most one exception. Hence the Gauss map omits at most 2 points. This contradiction shows that D must be the unit disk. In view of this we are reduced to the following:

(**)--Let f, g be holomorphic functions on the unit disk D, And $|f|>0$. Suppose that for six distinct complex numbers $a_1, a_2, \cdots, a_6$ the equation $g(z)-a_i$ has no solution $(i=1,2,\cdots,6)$. Then the mectric $\lambda^2|dz|^2$ on D is not complete, where $\lambda=|f|(1+|g|^2)/2$.

Proof. Suppose that the metric is complete and consider the function $h=f^{-2/p}g'\Pi_{i=1}^{6}(g-a_i)^{-\alpha}$, where $10/11\leqslant\alpha<1$ is as in the lemma 2 and $p=5/(6\alpha)$. $f^{-2/p}$ is well--defined because $|f|>0$. Since $g_{ij}=\lambda^2\delta_{ij}$ and $g^{ij}=\lambda^{-2}\delta_{ij}$, the Laplace-Beltrami operator Δ is given by the formula: for $k\in C^\infty(M)$,
$$\Delta(k)=\sqrt{g}\sum_{i,j}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,g^{ij}\frac{\partial k}{\partial x^j}\right)=\frac{1}{\lambda^2}\left(\frac{\partial^2 k}{\partial x^2}+\frac{\partial^2 k}{\partial y^2}\right)=\frac{4}{\lambda^2}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}k,$$
where $z=x+iy$. Let $u=|h|=(h\bar h)^{1/2}$. Then $\Delta\log u=(\Delta\log h+\Delta\log\bar h)/2=0$ almost everywhere, because of the fact that $g'=0$ on a set of measure zero and at every point where $g'\neq 0$, $\log h$ has a holomorphic branch in a neighborhood of that point. We assert that $u\notin L^p(D)$. Indeed, if u is a (necessarily nonzero) constant, this follows from the fact that complete simply connected surfaces of non-positive curvature have infinite volume. If u is not constant this follows from the lemma 3. Since the area element is $\lambda^2 dx\,dy$ and $\lambda=|f|(1+|g|^2)/2$, the condition $u\notin L^p(D)$ can be written $\frac{1}{4}\int_D|g'|^p(1+|g|^2)\Pi_{i=1}^{6}|g-a_i|^{-p\alpha}dx\,dy=\infty$. The contradiction will be achieved by showing that this integral is actually finite. Let $D_j=\{z\in D:|g(z)-a_j|\leqslant s\}$, where $0<s<\frac{1}{4}\min\{|a_i-a_k|:i\neq k;\ i,k=1,2,\cdots,6\}$. Then for $i\neq j$, $D_i\cap D_j=\phi$. Let $D^c=D-\cup_{j=1}^{6}D_j$.

Denoting by $H(z)$ the integrand of the last integral, we have
$$\int_D H\,dx\,dy=\sum_{j=1}^{6}\int_{D_j}H\,dx\,dy+\int_{D^c}H\,dx\,dy.$$

(1) On each D_j, since $|g|\leqslant|a_j|+s$ and $|g-a_i|^{-p\alpha}\leqslant(3s)^{-p\alpha}$ for each $i(\neq j)$, we have an estimate $H\leqslant C(|g'|^p/|g-a_j|^{p\alpha})$, we may also assume $s<1$, hence $|g-a_j|^{2-\alpha}<|g-a_j|^\alpha$. Thus $2|g-a_j|^\alpha>|g-a_j|^\alpha+|g-a_j|^{2-\alpha}$,

so that $|g'|^p/|g-a_j|^{p\alpha} \leqslant 2^p|g'|^p/(|g-a_j|^\alpha+|g-a_j|^{2-\alpha})^p$.
Hence $\int_{D_j} H\,dx\,dy < \infty$ by the lemma 2.

(2) On D^c, since $(1+|g|^2)^2|g-a_6|\Pi_{j=1}^{6}|g-a_j|^{-p\alpha} \leqslant B$ for some B, $H \leqslant B|g'|^p|g-a_6|^{-1}$. (note that $p\alpha=5/6$.) The fact that $1/p \geqslant \alpha$ and $10/11 \leqslant \alpha$ implies that $C_1|g-a_6|^{1/p} \geqslant |g-a_6|^\alpha$ and $C_2|g-a_6|^{1/p} \geqslant |g-a_6|^{2-\alpha}$ for some constants C_1, C_2. Hence $C|g-a_6|^{1/p} \geqslant (|g-a_6|^\alpha+|g-a_6|^{2-\alpha})$ for some C, so that $H \leqslant C'|g'|^p/(|g-a_6|^\alpha+|g-a_6|^{2-\alpha})^p$ on D^c for some C'. Therefore $\int_{D^c} H\,dx\,dy < \infty$ by the lemma 2.

NOTE. The problem of determining the exact size of the image under the Gauss map of complete regular minimal surfaces in R^3 is still unsolved, although many mathematicians have tried to solve it.

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