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On the Tensor Products of C^*-Algebras

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In the theory of von Neumann algebras, the classification of factors is an important problem. This is to classify factors to non $*$-isomorphic classes and is accomplished by finding algeblaic invariants.

Our study here is the classification of factors constructed from a product state of an infinite tensor product of C^*-algebras considered by E. Störmer [6], when the resulting factors are of type Ⅲ. Generally it is not known whether the tensor product of factors with type Ⅲ_0 is of type Ⅲ_0. But as an application of our classification, we can prove that the tensor product of infinitely many factors with type Ⅲ_0 is not of pe Ⅲ_0.

Also our result contains the famous uncontablly many non $*$-isomorphic hyperfinite factors with type Ⅲ introduced by R. T. Powers [5].

Ⅰ. DEFINITIONS AND LEMMAS

We refer the terminology, technical term and the results with respect to C^*-algebras and von Nenmann algebras, [1, 4], infinite tensor products of C^*-algebras and von Neumann algebras, [2, 4].

If A is a C^*-algebra and s is a state of A, then (π_s, H_s, h_s) denotes the cyclic $*$-representation of A constructed by s such that $s(x) = (\pi_s(x) h_s, h_s)$ for $x \in A$ where π_s is the $*$-representation of A on the Hilbert space H_s and h_s is the generating unit vector in H_s.

Let A_i : $i = 1, 2, \cdots$ be C^*-algebras with identity and s_i be a state of A_i. Then we can define $B = \overset{\infty}{\underset{i=1}{\otimes}} {}^* A_i$, $t = \overset{\infty}{\underset{i=1}{\otimes}} s_i$, the infinite tensor product of A_i and of s_i. B is a C^*-algera and t is a state of B. B is to be considered as the inductive limit of the finite tensor product $\overset{n}{\underset{i=1}{\otimes}}{}^* A_i$, $n = 1, 2, \cdots$, where the norm is the smallest C^*-crossnorm.

Let H_i, $i = 1, 2, \cdots$ be Hilbert spaces and h_i be an unit vector in H_i, and M_i be a von Neumann algebra acting on H_i. $\overset{\infty}{\underset{i=1}{\otimes}}{}^{(h_i)} H_i$ denotes the incomplete infinite tensor product Hilbert space with reference vectors (h_i) and $\overset{\infty}{\underset{i=1}{\otimes}} M_i$ denotes the von Neumann algebra acting on $\overset{\infty}{\underset{i=1}{\otimes}}{}^{h_i} H_i$ which is an infinite tensor product of M_i, $i=1,2,\cdots$.

It is well-known that H_t is isometrically isomorphic to $\overset{\infty}{\underset{i=1}{\otimes}}{}^{(h_i)} H_{s_i}$ and $\pi_t(B)''$ is spatially $*$-isomorphic to $\overset{\infty}{\underset{i=1}{\otimes}} \pi_{s_i}(A_i)''$. Moreover if $\pi_{s_i}(A_i)'' : i=1,2,\cdots$ are factors, then $\pi_t(B)''$ is a factor. cf. [2].

If A is a C^*-algebra and s is a state of A, then we call s a factor state if $\pi_s(A)''$ is a factor. When s is a factor state of A, s is said to be of type X if $\pi_s(A)''$ is of type X $X=I_n$, $n=1,2,\cdots$, II_1, II_∞, III_λ, $0\leq\lambda\leq 1$.

If A is a C^*-algebra and G is a group, we say A is asymptotically abelian with respect to G if there exists a representation $g\longrightarrow\alpha_g$ of G as $*$-automorphisms of A and a sequence $(g_n : n=1,2,\cdots)$ in G such that for $x,y\in A$

$$\| \alpha_{g_n}(x)y-y\alpha_{g_n}(x) \| \longrightarrow 0 \text{ when } n \text{ tends to } \infty.$$

Let G be a group of finite permutations of positive integer *i.e.*, $g\in G$ is an one-to-one map of positive integers onto itself which leaves all but a finite number of integers fixed. Then if $A_i=A$ is a C^*-algebra with an identity $i=1,2,\cdots$, and $B=\overset{\infty}{\underset{i=1}{\otimes}}{}^* A_i$, G acts on B by $\alpha_g(\sum\otimes x_i)=\sum\otimes x_{g(i)}$ $g\in G$, and B is asymptotically abelian with respect to G. And also $t=\otimes s_i$ is a G-invariant strongly clustering state of B, where $s_i=s$ is a state of A, $i=1,2,\cdots$ cf. [6].

If s is a state of a C^*-algebra A with an identity, E. Störmer introduced the spectrum of s, denoted by Spec s, is the set of non negative real numbers u such that, for a given $\varepsilon>0$, there exists an $x\in\pi_s(A)''$ for which $\| xh_s \| = 1$ and

$$| u(yxh_s, h_s)-(xyh_s, h_s) | < \varepsilon \| yh_s \|$$

for all $y\in\pi_s(A)$.

We denote the support of a state s of A by $E_s=[\pi_s(A)'h_s]$, where $[K]$ is the projection onto the closed linear span generated by K in H_s. h_s is then a cyclic and separating vector for the reduction $\pi_s(A)''_{E_s}$.

Let $\bar{\Delta}_s$ be the modular operator for h_s relative to $\pi_s(A)''_{E_s}$. We define Δ_s as an (unbounded) operator on H_s such that

$$\Delta_s=\begin{cases}\bar{\Delta}_s & \text{on } E_sH_s\\ 0 & \text{on } (1-E_s)H_s.\end{cases}$$

LEMMA 1 ([7], Theorem 2.3, Corollary 2.4) Let A be a C^*-algebra with an identity and s be a state of A. Then

Spec $s = Sp\ \Delta_s$, where $Sp\ \Delta_s$ is the spectrum of Δ_s in the usual sense. also we have that

(1) Spec $s = \{1\}$ iff s is a trace of A.

(2) Spec $s = \{0, 1\}$ iff the vector state $w_{h_s} = (\cdot h_s, h_s)$ is *a* trace on $\pi_s(A)'$ but s is not a trace of A.

If M is a von Neumann algebra, we define $S'(M) = \cap Sp\Delta_w$ where w runs through the set of all normal states of M. Then we have the following result in [7] Theorem 3. 1, Corollary 4. 2.

LEMMA 2 Let A be a C^*-algebra which is asymptotically abelian with respect to a group G and s be a G-invariant strongly clustering state of A. Then

(1) Spec $s - \{0\}$ is a closed subgroup of the multiplicative group of positive real numbers R^*_+ , and

(2) $S'(\pi_s(A)'') = $ Spec s.

If M is a factor, A. Connes has defined the algebraic invariant $S(M) = \cap Sp\ \Delta_w$ where w runs through the set of all normal faithful states of M. cf [1].

LEMMA 3 ([1] P. 188) Let M be a factor and E be a non zero projection of M. Then $S(M) = S(M_E)$.

2. MAIN THEOREM

E. Störmer considered the product state of an infinite tensor product of a C^*-algebra as its copy. He obtained the following result in [6] :

THEOREM 4 Let $A_i = A$ be a C^*- algebra with an identity and $B = \bigotimes_{i=1}^{\infty}{}^* A_i$. Let s be a factor state of A. Then $t = \otimes s$ is a factor state of B. Moreover the followings are true.

(1) t is of type I_1 iff s is a homomorphism of A.

(2) t is of type I_∞ iff s is a pure state of A but is not a homomorphism.

(3) t is of type II_1 iff s is a trace of A but is not a homomorphism.

(4) t is of type II_∞ iff w_{h_s} is a trace of $\pi_s(A)'$ but s is neither a pure state nor a trace of A.

(5) t is of type III iff w_{h_s} is not a trace of $\pi_s(A)'$.

We can classify the case (5) of Theorem 4 more refinely using the Connes' invariant S. cf. [1].

THEOREM 5 Let A, s, B, t be as same as in Theorem 4. We suppose the case (5) of Theorem 4 occurs. Then we have

(1) t is of type III_λ : $0<\lambda<1$ iff Spec $s-\{0\}$ is a subset of a non trivial closed subgroup of the multiplicative group R^*_+. In this case

$$\log \lambda = \max \{\log \frac{u_1}{u_2} : u_1 < u_2,\ u_1, u_2 \in \mathrm{Spec}\ s - \{0\}\}.$$

(2) t is of type III_1 in all other cases.

The proof of Theorem 5 is based on the following lemmas.

LEMMA 6 Let A, s, B, t be as above. Then we have $S'(\pi_t(B)'') = S(\pi_t(B)'')$

PROOF By the definition, it is clear that $S'(\pi_t(B)'') \subset S(\pi_t(B)'')$. If w_{h_s} is faithful on $\pi_s(A)''$, then $w_{h_t} = \otimes w_{h_s}$ is faithful on $\pi_t(B)''$. cf [2]. And so $Sp\Delta_t =$ Spec $t = S'(\pi_t(B)'') \subset S(\pi_t(B)'') \subset Sp\Delta_t$, because of Lemma 2 and $\Delta_t = \bar{\Delta}_t$ in this case. Suppose w_{h_s} is not faithful. Then

$$S'(\pi_t(B)'') = Sp\Delta_t = \{0\} \cup Sp\,\bar{\Delta}_t = \{0\} \cup S'(\pi_t(B)''_{E_t}) = \{0\} \cup S(\pi_t(B)'')$$

because of faithfullness of w_{h_t} on $\pi_t(B)''_{E_t}$ and $E_t \in \pi_t(B)''$. Hence we can use Lemma 3 and the faithful case. *Q. E. D.*

LEMMA 7 Let A_i : $i=1,2,\cdots$ be C^*-algebras with an identity 1_i and s_i be a state of A_i. Put $B = \overset{\infty}{\underset{i=1}{\otimes}}{}^* A_i$ and $t = \otimes s_i$. Then we have $\overset{\infty}{\underset{i=1}{\cup}}$ Spec $s_i \subset$ Spec t and $(\overset{\infty}{\underset{i=1}{\cup}}$ Spec $s_i - \{0\})^\perp \subset ($Spec $t - \{0\})^\perp$ where $P^\perp$ denotes the annihilator of P in the dual of R^*_+.

PROOF We can identify h_t with $\otimes h_{s_i}$, and $\pi_t(B)''$ is identified with $\overset{\infty}{\underset{i=1}{\otimes}} \pi_{s_i}(A_i)''$ as in 1. For fixed i, let u be in Spec s_i. Then for given $\varepsilon > 0$ there exists an $x_i \in \pi_{s_i}(A_i)''$ with $\| x_i h_{s_i} \| = 1$ such that, for all $y \in \pi_{s_i}(A_i)$,

$$(*)\ | u w_{h_{s_i}}(y x_i) - w_{h_{s_i}}(x_i y) | < \varepsilon \| y h_{s_i} \| .$$

Put $\bar{x}_i = 1_1 \otimes \cdots \otimes 1_{i-1} \otimes x_i \otimes 1_{i+1} \otimes \cdots \in \pi_t(B)''$. Then $\| \bar{x}_i h_t \| = 1$.

For $z \in B$ and for an arbitrary $\delta > 0$, there exists $z' \in \overset{m}{\underset{i=1}{\odot}} A_i \otimes 1$ for a certain integer m, where $\odot$ denotes the algebraic tensor product, such that $\| z - z' \| < \delta$. Then we have

$$| u w_{h_t}(y \bar{x}_i) - w_{h_t}(\bar{x}_i y) | \leqq u | w_{h_t}((y-y')\bar{x}_i) | + | u w_{h_t}(y' \bar{x}_i) - w_{h_t}(\bar{x}_i y') | + | w_{h_t}(\bar{x}_i (y'-y)) | \leqq (u \| x_i \| + 1)\delta + | u w_{h_t}(y' \bar{x}_i) - w_{h_t}(\bar{x}_i y') |$$

where $y = \pi_t(z)$ and $y' = \pi_t(z')$. We can set $y' = \sum_{p=1}^{n} (\overset{m}{\underset{q=1}{\otimes}} y_{pq}) \otimes 1_{m+1} \otimes \cdots$ in $\overset{m}{\underset{q=1}{\odot}} \pi_{s_q}(A_q) \otimes 1$. Then, we have

$$u w_{h_t}(y' \bar{x}_i) - w_{h_t}(\bar{x}_i y')$$

$$= u\sum_{p}\prod_{q\neq i}(y_{pq}h_{s_q}, h_{s_q})\, w_{h_{s_i}}(y_{p_i}x_i) - \sum_{p}\prod_{q\neq i}(y_{pq}h_{s_q}, h_{s_q})\, w_{h_{s_i}}(x_i y_{p_i})$$

$$= uw_{hsi}\Big(\big(\sum_{p}\prod_{q\neq i}(y_{pq}h_{sq}, h_{sq})\, y_{pi}\big)x_i\Big) - w_{hsi}\Big(x_i\big(\sum_{p}\prod_{q\neq i}(y_{pq}h_{sq}, h_{sq})\, y_{pi}\big)\Big).$$

By $(*)$ $\quad | uw_{h_t}(y'\bar{x}_i) - w_{h_t}(\bar{x}_i y') |^2 \leqq \varepsilon^2 \| \sum_{p}\prod_{q\neq i}(y_{pq}h_{s_q}, h_{s_q})\, y_{pi}h_{s_i} \|^2$

$$= \varepsilon^2 \sum_{p,r}\prod_{q\neq i}(y_{pq}h_{s_q}, h_{s_q})\,(h_{s_q}, y_{rq}h_{s_q})\,(y_{pi}h_{s_i}, y_{ri}h_{s_i})$$

$$= \varepsilon^2 \sum_{p,r}\Big(\big(\bigotimes_{q\neq i} y_{pq}h_{s_q}\big)\otimes\big(\bigotimes_{q\neq i} h_{s_q}\big)\otimes y_{pi}h_{s_i},\ \big(\bigotimes_{q\neq i} h_{s_q}\big)\otimes\big(\bigotimes_{q\neq i} y_{rq}h_{s_q}\big)\otimes y_{ri}h_{s_i}\Big)$$

$$\leqq \varepsilon^2 \| \sum_{p}\big(\bigotimes_{q\neq i} y_{pq}h_{s_q}\big)\otimes\big(\bigotimes_{q\neq i} h_{s_q}\big)\otimes y_{pi}h_{s_i} \| . \| \sum_{r}\big(\bigotimes_{q\neq i} h^{sq}\big)\otimes\big(\bigotimes_{q\neq i} y_{rq}h_{s_q}\big)\otimes y_{ri}h_{s_i} \| .$$

But $\quad \| \sum_{p}\big(\bigotimes_{q\neq i} y_{pq}h_{s_q}\big)\otimes\big(\bigotimes_{q\neq i} h_{s_q}\big)\otimes y_{pi}h_{s_i} \|^2 = \| \sum_{p}\bigotimes_{q} y_{pq}h_{s_q} \|^2 = \| y'h_t \|^2,$

since $\quad \| h_{s_q} \| = 1$, and $\pi_{sq}(1_q)h_{s_q} = h_{s_q}$ for $q > m$. Analogously

$\| \sum_{r}\big(\bigotimes_{q\neq i} h_{s_q}\big)\otimes\big(\bigotimes_{q\neq i} y_{rq}h_{s_q}\big)\otimes y_{ri}h_{s_i} \|^2 = \| y'h_t \|^2$. Hence we have

$| uw_{h_t}(y'\bar{x}_i) - w_{h_t}(x_i y') | \leqq \varepsilon \| y'h_t \|$

and $\quad | uw_{h_t}(y\bar{x}_i) - w_{h_t}(x_i y) | \leqq \delta(u \| x_i \| + 1) + \varepsilon \| y'h_t \| \leqq \delta(u \| x_i \| + 1 + \varepsilon)$
$+ \varepsilon \| yh_t \|$.

Since δ is arbitrary, we obtain $u \in \mathrm{Spec}\ t$.

Let v be in $(\bigcup_p \mathrm{Spec}\ s_p - \{0\})^\perp$. Since $\mathrm{Spec}\ s_p - \{0\} = Sp\Delta_{s_p}$, we have $\Delta_{s_p}^{iv} = E_{s_p}$. Also we know the equality $\Delta_t = \bigotimes_{p=1}^{\infty} \Delta_{s_p}$, cf [3]. But $E_t = \bigotimes_{p=1}^{\infty} E_{s_p}$, hence $\Delta_t^{iv} = \bigotimes \Delta_{sp}^{iv} = \bigotimes E_{sp} = E_t$. It means that t is in $(Sp\Delta_t)^\perp$. But $Sp\Delta_t = \mathrm{Spec}\ t - \{0\}$. Q. E. D.

PROOF of Theorem 5. By Lemma 2 and Lemma 7, Spec $t - \{0\}$ is the closed subgroup generated by Spec $s - \{0\}$ in R_+^*. By lemma 2 and 6, we have $S(\pi_t(B)'' - \{0\} = \mathrm{Spec}\ t - \{0\}$. Then the condition : $S(\pi_t(B)'') - \{0, 1\}$, *i. e. t* is of type III_0, cannot occur. If this condition occurs, it is equivalent to Spec $s = \{1\}$ or $\{0, 1\}$. By the way this is equivalent that t is of type II_1 or of type II_∞ by Lemma 1 and Theorem 4. This contradiction shows that t cannot be of type III_0.

If t is of type III , then the above consideration shows that Spec $s - \{0\}$ is a subset of $\{\lambda^n : n = 0, \pm 1, \pm 2, \cdots\cdots\}$ and that $\log\lambda = \max\ \{\log \frac{u_1}{u_2} : u_2 > u_1,$
$u_1, u_2 \in \mathrm{Spec}\ s - \{0\}\}$.

The converse is obviously true. Other cases are those for t to be of type III_1.

Q. E. D.

3. EXAMRLES AND APPLICATIONS

Example 8 R. T. Powers [5] has introduced uncountably many non $*$-isomorphic hyperfinite factors with type III, so called Powers' factors R_λ : $0 < \lambda < 1$. R_λ is constructed by the infinite tensor product $\otimes s$ where s is the faithful state of $M_2(C)$, 2×2 matrices.

Put $A = M_2(C)$ and let s be a faithful state on $M_2(C)$. Then there is a positive matrix D_s in A such that $s=tr(D_s\cdot)$, where tr is a normalized trace in $M_2(C)$. Let λ_1, λ_2 be eigenvalues of D_s. Then we have that $\lambda_1+\lambda_2=1$, $\lambda_1, \lambda_2>0$. Take orthogonal unit vectors h_1, h_2 in C^2 such that $s=\lambda_1 w_{h_1}+\lambda_2 w_{h_2}$.

We shall show that

$$\text{Spec } s=\{\frac{\lambda_2}{\lambda_1}, \frac{\lambda_1}{\lambda_2}\}.$$

Suppose that u is in Spec s. Then for given $\varepsilon>0$, there exists $x\in A$ with $s(x^*x)=1$ such that for all $y\in A$ $|us(yx)-s(xy)|<\varepsilon s(y^*y)$. Let $\{e_{ij} : i,j=1,2\}$ be matrix units of A such that $e_{ij}^*e_{ij}=[h_i]$ and $e_{ij}e_{ij}^*=[h_j]$. Now we substitute e_{ij} in place of y of the above inequality. We have $|u\lambda_i-\lambda_j|\cdot|(xh_j, h_i)|$ $<\varepsilon\lambda_i^{\frac{1}{2}}$.

If u was not in $\{\frac{\lambda_j}{\lambda_i}: i,j=1,2\}$, this inequality shows $|(xh_j, h_i)|^2<\varepsilon^2\lambda_i/(u\lambda_i-\lambda_j)^2$.

But for arbitrary $\varepsilon>0$,

$$1=s(x^*x)=\lambda_1\|xh_1\|^2+\lambda_2\|xh_2\|^2=\sum_{i,j}\lambda_j|(xh_j, h_i)|^2<\varepsilon^2\sum_{i,j}\lambda_i\lambda_j(u\lambda_i-\lambda_j)^{-2}.$$

This shows that Spec $s\subset\{\frac{\lambda_2}{\lambda_1}, \frac{\lambda_1}{\lambda_2}\}$. Conversely for $u=\frac{\lambda_i}{\lambda_j}$, it is sufficient to set $x=\lambda_j^{-\frac{1}{2}}e_{ji}$. Our considering object contains the Powers' fators.

Analogously if s is a faithful state of $M_n(C)$, $n\geqq 3$ and eigenvalues of s are λ_j : $j=1,2,\cdots,n$, $\lambda_j>0$, $\sum_{j=1}^{n}\lambda_j=1$, then we have Spec $s=\{\lambda_j\lambda_i^{-1} : i,j=1,2,\cdots n\}$.

COROLLARY 9 Let $A=M_n(C)$ and s be a state of A. We preserve the notation in Theorem 4. Then we have the followings.

(1) t is of type III_λ, $0<\lambda<1$ iff $\log u_2/\log u_1$ is a rational number for all $u_1, u_2\in$ Spec $s-\{0\}$.

(2) t is of type III_1 iff there are u_1 and u_2 in Spec $s-\{0\}$ such that $\log u_2/\log u_1$ is irrational.

PROOF For $u_1, u_2>0$, suppose there are integers n and m such that $u_1^n=u_2^m$. Then $\log u_2/\log u_1=n/m$. Also Spec s is a finite set. *Q. E. D.*

In general it has not been known whether finite tensor products of factors with type III_0 are of type III_0. But we have the following.

COROLLARY 10 Let M be a factor with type III_0. Then $\overset{\infty}{\otimes}M$ cannot be of type III_0.

PROOF Let s be a non trivial faithful normal state of M. Then we can identify M with $\pi_s(M)$. The desired conclusion followes from Theorem 5. *Q. E. D.*

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