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Some Properties of the Group Rings

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Let K be a field and G a multiplicative group.

The group ring $K(G)$ is an extension ring of $K(H)$ for any subgroup H of G. Is $K(G)$ an integral extension ring? We shall prove that $K(G)$ is an integral extension ring of $K(H)$ if H is a subgroup of finite index in G.

P. Hall have proved that (1) if G is a polycyclic-by-finite group then $K(G)$ is a right Noetherian ring and (2) if G is a solvable group and $K(G)$ is a right Noetherian ring then G is polycyclic [4].

For any normal subgroup N of a polycyclic-by-finite group G, G/N is polycyclic-by-finite. Therefore, we can see that if $K(G)$ is a right Noetherian ring for a solvable group G then $K(G/N)$ is a right Noetherian ring. But the converse is not true. We shall prove that the converse is true in case that N is a finite normal subgroup of G.

And we shall prove that G is a finite group if and only if $K(G)$ is a right Noetherian ring and right perfect ring.

Theorem 1. Let G be a group. If H is a subgroup of finite index in G then $K(G)$ is an integral extension ring of $K(H)$.

Proof. Since $Y=\{y\in G \mid G=\cup Hy\}$ is finite, let $Y=\{e=y_1, y_2, \cdots, y_n\}$. For $\alpha=\sum\alpha(g)g\in K(G)$, We can write as following

$$\alpha=\sum\alpha(g)g=\sum_{g\in Hy_1}\beta(g)g+\cdots+\sum_{g\in Hy_n}\gamma(g)g$$
$$=\sum_{g\in Hy_1}\beta(g)gy_1^{-1}y_1+\cdots+\sum_{g\in Hy_n}\gamma(g)gy_n^{-1}y_n.$$

Therefore, since $\sum\beta(g)gy_1^{-1}, \cdots, \sum\gamma(g)gy_n^{-1}\in K(H)$, $K(G)$ is a left $K(H)$-module with a finite generator set Y. Hence

$$\alpha y_i=\gamma_{i1}y_1+\cdots+\gamma_{in}y_n \quad (\gamma_{ij}\in K(H)).$$

Let M be a $n\times n$ matrix (γ_{ij}). Then $|M-\alpha I|y_1=0$. Therefore

$$|M-\alpha I|=0.$$

Thus, $\alpha \in K(G)$ is a root of $|M-xI|$ of $K(H)[x]$.

Corollary. If G is finite, then $K(G)$ is an integral extension ring of K. Thus, if H is a normal subgroup of finite index in a group G then $K(G/H)$ is an integral extension ring of K. ∎

Let G be a polycyclic-by-finite group. Then there is a submodule series

$$\{e\}=G_0 \Delta G_1 \Delta \cdots \Delta G_n = G$$

with quotient that are either cyclic or finite. Of course, G_0 has a characteristic subgroup of finite index that is poly-{infinite, cyclic}. Suppose that G_i has a characteristic subgroup H_i of finite index that is poly-{infinite, cyclic}. Since $G_i \Delta G_{i+1}$ and H_i is characteristic in G_i, $H_i \Delta G_{i+1}$. If G_{i+1}/G_i is finite then G_{i+1} has a normal poly-{infinite, cyclic} subgroup H_i of finite index. If G_{i+1}/G_i is infinite cyclic, then G_{i+1} has a finite normal subgroup G_i/H_i with $G_{i+1}/H_i/G_i/H_i \cong G_{i+1}/G_i$ in finite cyclic. If

$$G_{i+1}/H_i = <G_i/H_i, gH_i>$$

then g^tH_i for some $t \geqq 1$ certainly centralizes G_i/H_i and gH_i. Hence g^tH_i is a central element in G_{i+1}/H_i. This implies that $<g^tH_i>$ is a normal infinite cyclic subgroup of G_i/H_i with

$$|G_{i+1}/H_i : <g^tH_i>| < \infty.$$

Therefore, the inverse image M of g^tH_i in G_{i+1}/H_i is a normal poly-{infinite, cyclic} subgroup of G_{i+1} of finite index. Since G_{i+1} is a finitely generated group, G_{i+1} has only finitely many subgroup of index equal to the index of H. Let H_{i+1} be their intersection. Then H_{i+1} is a characteristic subgroup of G_{i+1} of finite index. Since $H_{i+1} \leqq H$ and the class of poly-{infinite, cyclic} is closed under taking subgroups, H_{i+1} is also poly-{infinite, cyclic}.

By induction step, the polycyclic-by-finite group G has a normal poly-{infinite, cyclic} subgroup of finite index. Therefore, if G is a polycyclic-by-finite group then G has a subgroup H such that $K(G)$ is an integral extension ring of $K(H)$.

If G is finite, then $K(H \times G)$ is an integral extension ring of $K(H)$.

An infinite dihedral group $G=<x, y \mid y^2=1, y^{-1}xy=x^{-1}>$ has a normal infinite cyclic subgroup $<x>$ of index 2. Therefore, $K(G)$ is an integral extension ring of $K(<x>)$.

Lemma 1. (P. Hall). Let G be a solvable group. If K (G) is a right Noetherian ring then G is a polycyclic group.

Lemma 2. (P. Hall). Let G be a polycyclic-by-finite group. Then $K(G)$ is a right Noetherian ring.

From lemma 1, and 2, we have the following result.

Theorem 2. Let G be a solvable group and N a finite normal subgroup of G. Then if $K(G)$ is a right Noetherian ring then $K(G/N)$ is a right Noetherian ring and the converse is true.

Proof. If $K(G)$ is a right Noetherian ring then G is a polycyclic group by lemma 1. Hence, G/N is a polycyclic group. Therefore, by lemma 2, $K(G/N)$ is a right Noetherian ring. Conversely, Let $K(G/N)$ be a right Noethring. Since G is solvable, G/N is solvable. Therefore, G/N is a polycyclic group by lemma 1. Hence, there is a subnormal series

$$\{N\} \Delta H_1/N \Delta H_2/N \Delta \cdots \Delta H_r/N = G/N$$

with each factor group cyclic. Since $H_{i+1}/N/H_i/N \cong H_{i+1}/H_i$ is cyclic, we obtain a subnormal series

$$\{e\} \Delta N \Delta H_1 \Delta \cdots \Delta H_r = G$$

with each H_{i+1}/H_i cyclic, $N/\{e\}$ finite and H_i/N cyclic. Therefore, G is a polycyclic-by-finite group. Hence, $K(G)$ is a right Noetherian ring.

Lemma 3. (Bovid-Mihovski). Let e be a central idempotent in $K(G)$. Then $<$ Supp $e>$ is a finite normal subgroup of G.

We have the following from lemma 3.

Corollary. Let G be a solvable group and e a central idempotent element in $K(G)$. If $K(G)$ is a right Noetherian ring then $K(G/<\text{Supp } e>)$ is a right Noetherian ring and the converse is true.

Definition. Let G be a group. $x \varepsilon G$ is called a ***local quasicentral element*** (or ***local QC-element***) of G iff $<x>$ is a normal subgroup of G.

Theorem 3. Let G be a solvable group and $x \varepsilon G$ a local QC-element of G. If $K(G)$ is a right Noetherian ring then $K(G/<x>)$ is a right Noetherian ring and the converse is true.

Proof. See the proof of Theorem 2. Note that if $K(G/\langle x\rangle)$ is a right Noetherian ring then G is a polycyclic group. ▮

Lemma 4. (Connell). The group ring $K(G)$ is a right Artinian if and only if G is finite.

Lemma 5. (Woods. Renault). The group ring $K(G)$ is a perfect ring if and only if G is finite.

Theorem 4. The group ring $K(G)$ is right Noetherian and right perfect if and only if G is finite.

Proof. Let G be finite. Then G is polycyclic-by-finite. Therefore $K(G)$ is right Noetherian. And by lemma 5, $K(G)$ is perfect.

Suppose that $K(G)$ is right Noetherian and right perfect. Since $K(G)$ is right Noetherian, each of the right $K(G)$-modules $K(G)/JK(G)$. $JK(G)/JK(G)^2, \cdots$ is finitely generated, and $K(G)/JK(G)$, $JK(G)/JK(G)^2, \cdots$ are right $K(G)/JK(G)$-module. Since $K(G)$ is right perfect, $K(G)/JK(G)$ is a semisimple ring. Thus, each of the right $K(G)$-module $K(G)/JK(G)$, $JK(G)/JK(G)^2, \cdots$ is a finite direct sum of simple modules and hence has a composition series for n. Hence $K(G)/JK(G)^n$ is right artinian. Since $JK(G)$ is right T-nilpotent, $JK(G)^n=0$. Therefore, $K(G)$ is right Artinian. By lemma 4, G is finite. ▮

Thus, follows are equivalent ;

(a) G is a finite group.

(b) The group ring $K(G)$ is a right Artinian ring.

(c) The group ring $K(G)$ is a perfect ring.

(d) The group ring $K(G)$ is a right Noetherian ring and right perfect ring.

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