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Some Properties of Complex Grassmann Manifolds

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§ 1. Abstract

The hermitian structures on complex manifolds have been studied by several mathematicians ([1], [2], and [3]), and the Kähler structure on hermitian manifolds have been so much too ([6], [12], and [15]). There has been some gradual progress in studying the invariant forms on Grassmann manifolds ([17]).

The purpose of this dissertation is to prove the Theorem 3.4 and the Theorem 4.7, with relation to the nature of complex Grassmann manifolds.

In § 2, in order to prove the Theorem 4.7 which will be explicated further in § 4, the concepts of the hermitian structure, connection and curvature have been defined, and the characteristic nature about these were proved. (Proposition 2.3, 2.4, 2.9, 2.11, and 2.12)

Two characteristics were proved in § 3. They are almost not proved before: particularly, we proved the Theorem 3.3:

$$G_k(C^{n+k}) = \frac{GL(n+k, C)}{GL(k, n, C)} = \frac{U(n+k)}{U(k) \times U(n)}$$

In § 4, we explained and proved the Theorem 4.7:

i) *Complex Grassmann manifolds are Kählerian.*

ii) *This Kähler form is π-fold of curvature form in hyperplane section bundle.*

Prior to this proof, some propositions and lemmas were proved at the same time. (Proposition 4.2, Lemma 4.3, Corollary 4.4 and Lemma 4.5).

§ 2. Preliminaries

Definition 2.1. Let M be a C^∞-differentiable manifold. A *complex vector bundle*

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$\pi: E \to M$ over M with rank q consists of a Hausdorff space E and a continuous map π satisfying the following conditions:

i) for $x \in M$, $E_x = \pi^{-1}(x)$ is a complex vector space of dimension q.

ii) for $x \in M$, there is an open neighborhood U of x and a homeomorphism

$$h: \pi^{-1}(U) \longrightarrow U \times C^q$$

such that

$$h(E_x) \subset \{x\} \times C^q,$$

and h^x defined by the composition

$$h^x: E_x \xrightarrow{\ h\ } \{x\} \times C^q \xrightarrow{\ \text{proj}\ } C^q$$

is a complex vector space isomorphism, where the pair (U, h) is called a ***local trivialization***.

Note that for two local trivializations (U_α, h_α) and (U_β, h_β) the map

$$h_\alpha \circ h_\beta^{-1}: (U_\alpha \cap U_\beta) \times C^q \longrightarrow (U_\alpha \cap U_\beta) \times C^q$$

induces a map

$$g_{\alpha\beta}: (U_\alpha \cap U_\beta) \longrightarrow GL(q, C)$$

where for $x \in U_\alpha \cap U_\beta$

$$g_{\alpha\beta}(x) = h_\alpha^x (h_\beta^x)^{-1}: C^q \longrightarrow C^q$$

i.e., $g_{\alpha\beta}(x) \in GL(q, C)$. The function $\{g_{\alpha\beta}\}$ are called the ***transition functions*** of the complex vector bundle $\pi: E \to M$.

By our definition, it is easy to prove that the transition functions $\{g_{\alpha\beta}\}$ satisfy the following compatibility conditions:

i) $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = I_q$ on $U_\alpha \cap U_\beta \cap U_\gamma$

ii) $g_{\alpha\beta} = I_q$ on U_α,

where the product is a matrix product and I_q is the identity matrix of rank q.

In general, a set $\{g_{\alpha\beta}\}$ of transition functions satisfying the above compatibility conditions determines only one complex vector bundle over M ([7]).

Let V be a complex vector space. An ***hermitian structure*** on V is a complex valued function

$$H: V \times V \longrightarrow C$$

such that

i) $H(\lambda_1 \xi_1 + \lambda_2 \xi_2, \eta) = \lambda_1 H(\xi_1, \eta) + \lambda_2 H(\xi_2, \eta)$

where ξ_1, ξ_2, $\eta \in V$ and $\lambda_1, \lambda_2 \in C$,

ii) for ξ, $\eta \in V$

$$H(\xi, \eta) = \overline{H(\eta, \xi)}$$

In particular, if for $\xi(\neq 0) \in V$

$$H(\xi, \xi) > 0$$

then this hermitian structure is said to be ***positive definite.***

Definition 2. 2 Let $E \to M$ be a complex vector bundle. An ***hermitian structure*** H on E is a C^∞-field of positive definite hermitian structure in the fibres of E. That is, for two C^∞ sections ξ and η, $H(\xi, \eta)$ is a complex valued C^∞-function. Furthermore, for each $x \in M$ and a C^∞-section ν the following hold:

i) $H(\lambda_1 \xi_x + \lambda_2 \nu_x, \eta_x) = \lambda_1 H(\xi_x, \eta_x) + \lambda_2 H(\nu_x, \eta_x)$

where $\xi_x = \xi(x)$ and λ_1, $\lambda_2 \in C$

ii) $H(\xi_x, \eta_x) = \overline{H(\eta_x, \xi_x)}$

iii) $H(\xi_x, \xi_x) \geqq 0$ and $H(\xi_x, \xi_x) > 0$ if $\xi_x \neq 0$

A complex vector bundle with an hermitian structure is called an ***hermitian vector bundle.***

Proposition 2. 3. *Every complex vector bundle* $\pi: E \to M$ *admits a hermitian structure.*

Proof. Since M is a differentiable manifold, it has a locally finite covering $\{U_\alpha\}$ such that $E \mid U_\alpha$ is trivial for each U_α. We assume that $\{e_1, \cdots\cdots, e_n\}$ is a frame defined on U_α. For two C^∞-sections ξ and η of the bundle $\pi: E \longrightarrow M$, we shall define

$$\langle \xi_x, \eta_x \rangle = \sum_{i=1}^{n} \xi^i(x) \, \overline{\eta}^i(x),$$

where $\xi_x = \sum_{i=1}^{n} \xi^i(x) e_i(x)$ and $\eta_x = \sum_{i=1}^{n} \eta^i(x) e_i(x)$

In this case

$\xi^i : M \longrightarrow C$ are C^∞-functions for $i = 1, \cdots\cdots, n$. Now let $\{\rho_\alpha\}$ be a C^∞-partition of unity subordinate to the covering $\{U_\alpha\}$.

We put

$$H(\xi_x, \eta_x) = \sum_\alpha \rho_\alpha(x) \langle \xi_x, \eta_x \rangle$$

then the function

$$H(\xi_x, \eta_x) = \sum_\alpha \rho_\alpha(x) \left(\sum_{i=1}^{n} \xi^i(x) \, \overline{\eta}^i(x) \right)$$

is a C^∞-function on M. It is easy to prove that H satisfies condition (i), (ii),

(iii) in Definition 2. 2. It follows that H is a hermitian structure on E. Q. E. D.

Let V be a real vector space and suppose that $J: V \longrightarrow V$ is a R-linear isomorphism such that $J^2 = -I_V$, where R is the set of all real numbers and I_V is the identity map of V. Then J is called a ***complex structure*** on V. We assume that V is a real vector space with a complex structure J and consider the complexification $V \otimes_R C$ of V. The R-linear mapping J extendeds to a C-linear mapping of $V \otimes_R C = V_C$ into itself by

$$\begin{array}{ccc} V_C & \longrightarrow & V_C \\ \Cup & & \Cup \\ v \otimes \alpha & \rightsquigarrow & J(v) \otimes \alpha \end{array}$$

then this C-linear mapping J has the property $J^2 = -1$. Here, we put

$$V^{1,0} = \{v - iJv \mid v \in V\}$$

$$V^{0,1} = \{v + iJv \mid v \in V\}$$

Then it is easily proved that

$$V_C = V \otimes_R C = V^{1,0} \oplus V^{0,1}$$

Note that for $v \otimes \alpha \in V \otimes_R C$

$$i\,(v \otimes \alpha) = v \otimes i\,\alpha, \quad \overline{v \otimes \alpha} = v \otimes \overline{\alpha}$$

Proposition 2. 4. *Under the above situation we have*

i) $V_J \cong V^{1,0}$ *which is a C-linear isomorphism*

ii) $V^{1,0} \cong V^{0,1}$,

where V_J is the complex vector space obtained from V by means of J.

Proof. i) We define

$$\begin{array}{ccc} f: V_J & \longrightarrow & V^{1,0} \\ \Cup & & \Cup \\ v & \rightsquigarrow & v \otimes 1 - J(v) \otimes i \end{array}$$

then the following diagram commutes

$$\begin{array}{ccc} V_J & \xrightarrow{\quad f \quad} & V^{1,0} \\ J \big\downarrow & \copyright & \big\downarrow i \\ V_J & \xrightarrow{\quad f \quad} & V^{0,1} \end{array}$$

Hence f is a C-linear map (Note that for $v \in V$, $iv = J(v)$). For each element $v \otimes \alpha - iJ(v) \otimes \alpha \in V^{1,0}$, since we have

$$\alpha(v\otimes 1-J(v)\otimes i)=v\otimes\alpha-iJ(v)\otimes\alpha$$

Therefore f is isomorphism.

ii) Define the conjugate linear mapping

$$\begin{array}{rcl} Q: V^{1,0} & \longrightarrow & V^{0,1} \\ & & \cup\!\!\!\!\!\cup \\ v & \rightsquigarrow & \bar{v} \end{array}$$

then by this mapping we see that $V^{1,0}\cong V^{0,1}$ (Q is called a ***conjugation***). Q. E. D.

Example 2.5. Let C^n be the usual Euclidean space of n-tuples of complex numbers $\{z^1, \cdots, z^n\}$. If we put

$$z^j=x^j+iy^j \quad (x^j,\ y^j\in R \quad 1\leqq j\leqq n)$$

then we can identify C^n with $R^{2n}=\{(x^1, y^1, \cdots\cdots, x^n, y^n) \mid z^j=x^j+iy^j\}$.

Define the mapping

$$J: R^{2n}\longrightarrow R^{2n}$$

by $\quad J(x_1, y_1, \cdots, x_n, y_n)=(-y_1, x_1, -y_2, x_2, \cdots, -y_n, x_n)$

then it is easy to prove that $J^2=-I_{R^{2n}}$. This J is called the ***standard complex structure*** on R^{2n}. The coset space $GL(2n, R)/GL(n, C)$ determines all complex structure on R^{2n} by

$$[A]\rightsquigarrow A^{-1}JA,$$

where $[A]$ is the equivalence class of $A\in GL(2n,\ R)$ ([11]).

For a real vector space V of rank n we shall consider the exterior algebras $\wedge V_C$, $\wedge V^{1,0}$ and $\wedge V^{0,1}$. Then there are natural injections

$$\begin{array}{l} \wedge V^{1,0} \\ \wedge V^{0,1} \end{array} \rightrightarrows \wedge V_C$$

Moreover, if we let $\overset{p,q}{\wedge} V$ be the subspace of $\wedge V_C$ generated by elements of the form $u\wedge v$, where $u\in\overset{p}{\wedge} V^{1,0}$ and $w\in\overset{q}{\wedge} V^{0,1}$, then we have the direct sum:

$$\wedge V_C=\sum_{r=0}^{2n}\sum_{p+q=r}\overset{p,q}{\wedge} V$$

Definition 2.6. Let M be a differentiable manifold of rank $2n$, and let $\pi: T(M)\longrightarrow M$ be the tangent bundle of M. If a differentiable vector bundle isomorphism

$$J: T(M)\longrightarrow T(M)$$

satisfies the condition which is that for each point $x\in M$

$$J_x: T_x(M)\longrightarrow T_x(M)$$

is a complex structure for $T_x(M)$ ($T_x(M)=\pi^{-1}(x)$ and $J_x=J \mid T_x(M)$), then J is called an ***almost complex structure*** for M. Also, (M, J) is called an ***almost complex manifold***.

By using Example 2.5 we can prove that every analytic complex manifold induces an almost complex structure on its underlying differentiable manifold ([16]), which is called the usual almost complex structure. In the sequel, by a complex manifold we mean an analytic complex manifold.

Let (M, J) be an almost complex manifold. Then, as before we have the following:

$$T(M)_c=T(M)^{1,0}\oplus T(M)^{0,1},\quad T(M)_J\cong T(M)^{1,0},\quad T(M)^{1,0}\cong T(M)^{0,1}\ \ ([9],\ [16]).$$

Furthermore, if we let $T^*(M)_c$ be the complexification of the cotangent bundle $T^*(M)$ of M, we have

$$T^*(M)_c=T^*(M)^{1,0}\oplus T^*(M)^{0,1},$$

and natural bundle injections

$$\begin{matrix}\wedge T^*(M)^{1,0} \\ \wedge T^*(M)^{0,1}\end{matrix} \rightrightarrows \wedge T^*(M)_c$$

Let $\overset{p,q}{\wedge} T^*(M)$ be the bundle over M whose fibre is $\overset{p,q}{\wedge} T^*_x(M)$, then its sections are the complex valued differential forms of type (p, q) on M. We denote the set of all sections of $\overset{p,q}{\wedge} T^*(M)$ by

$$\Gamma^{p,q}(M)=\Gamma(M,\ \overset{p,q}{\wedge} T^*(M)).$$

We also put

$$\Gamma^r(M)=\sum_{p+q=r}\Gamma^{p,q}(M).$$

Under the above situation we want to recall the exterior derivative

$$d: \Gamma^r(M) \longrightarrow \Gamma^{r+1}(M)$$

Let $\pi_{p,q}$ denote the natural projection operators

$$\pi_{p,q}: \Gamma^r(M) \longrightarrow \Gamma^{p,q}(M),$$

where $p+q=r$. In general we have

$$d: \Gamma^{p,q}(M) \longrightarrow \Gamma^{p+q+1}(M)=\sum_{r+s=p+q+1}\Gamma^{r,s}(M)$$

by restricting d to $\Gamma^{p,q}$. We define

$$\partial: \Gamma^{p,q}(M) \longrightarrow \Gamma^{p+1,q}(M)$$
$$\bar{\partial}: \Gamma^{p,q}(M) \longrightarrow \Gamma^{p,q+1}(M)$$

as compositions

$$\partial : \Gamma^{p,q}(M) \xrightarrow{\ d\ } \Gamma^{p+q+1}(M) \xrightarrow{\ \pi_{p+1,q}\ } \Gamma^{p+1,q}(M)$$
$$\bar{\partial} : \Gamma^{p,q}(M) \xrightarrow{\ d\ } \Gamma^{p+q+1}(M) \xrightarrow{\ \pi_{p,q+1}\ } \Gamma^{p,q+1}(M)$$

and

If $d=\partial+\bar{\partial}$ then the almost complex structure is said to be *integrable*. Since $d^2=0$, we have

$$\partial^2=\bar{\partial}^2, \quad \partial\bar{\partial}=-\bar{\partial}\partial$$

Proposition 2.7. *Let M be a complex manifold of dimension n. Then the usual almost complex structure on M is integrable.*

Proof. We denote the underlying differentiable manifold of M by M_0. If J is the usual almost complex structure on M_0, it is easy to prove that $T(M)$ is C-linear isomorphic to $T(M_0)_J$ as C-bundles. Therefore it follows that, as C-bundles,

$$T(M)\cong T(M_0)^{1,0}, \quad T^*(M)\cong T^*(M_0)^{1,0}, \quad (\text{see}\,[9])$$

A local coordinates $(z^1, \cdots\cdots, z^n)$ of M $\{dz^1, \cdots\cdots, dz^n\}$ is a local frame for $T^*(M)^{1,0}$. Hence, by (ii) of proposition 2.4 $\{d\bar{z}^1, \cdots\cdots, d\bar{z}^n\}$ is a local frame for $T^*(M_0)^{0,1}$. Put

$$z^j=x^j+iy^j \qquad (1\leqq j\leqq n),$$

then it follows that

$$dz^j=dx^j+idy^j, \quad d\bar{z}^j=d\bar{x}^j-id\bar{y}^j,$$

which gives

$$dx^j=\frac{1}{2}(dz^j+d\bar{z}^j), \quad dy^j=\frac{1}{2i}(dz^j-d\bar{z}^j) \quad (1\leqq j\leqq n).$$

Therefore each $s\in\Gamma^{p,q}(M)$ can be represented by

$$s=\sum_{\substack{1\leq i_1<\cdots<i_p\leq n\\ 1\leq j_1<\cdots<j_q\leq n}} a_{i_1\cdots i_p j_1\cdots j_q} dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q}$$

Hence we have

$$ds=\sum_{\substack{1\leq i_1<\cdots<i_p\leq n\\ 1\leq j_1<\cdots<j_q\leq n}} \Big(\sum_{j=1}^{n}\big(\frac{\partial}{\partial x^j}a_{i_1\cdots i_p j_1\cdots j_q}dx^j+\frac{\partial}{\partial y^j}a_{i_1\cdots i_p j_1\cdots j_q}dy^j\big)\wedge dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q}$$

$$=\sum_{j=1}^{n}\sum_{\substack{1\leq i_1<\cdots<i_p\leq n\\ 1\leq j_1<\cdots<j_q\leq n}} \frac{\partial}{\partial z^j}a_{i_1\cdots i_p j_1\cdots j_q}dz^j\wedge dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q}$$

$$+\sum_{j=1}^{n}\sum_{\substack{1\leq i_1<\cdots<i_p\leq n\\ 1\leq j_1<\cdots<j_q\leq n}} \frac{\partial}{\partial \bar{z}^j}a_{i_1\cdots i_p j_1\cdots j_q}d\bar{z}^j\wedge dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q}$$

Note that

$$\frac{\partial}{\partial z^j}=\frac{1}{2}\Big(\frac{\partial}{\partial x^j}-i\frac{\partial}{\partial y^j}\Big), \quad \frac{\partial}{\partial \bar{z}^j}=\frac{1}{2}\Big(\frac{\partial}{\partial x^j}+i\frac{\partial}{\partial y^j}\Big)$$

The first term is of type $(p+1, q)$, and thus

$$\partial=\sum_{j=1}^{n}\frac{\partial}{\partial z^j}dz^j,$$

and similarly

$$\bar{\partial}=\sum_{j=1}^{n}\frac{\partial}{\partial \bar{z}^j}d\bar{z}^j$$

Therefore $d=\partial+\bar{\partial}$. Q. E. D.

Let M be a (C^∞)-differentiable manifold of rank n, and let $\pi: E\longrightarrow M$ be a q-dimensional complex bundle over M. For the cotangent bundle $T^*(M)$ of M we denote by $\Gamma(E)$ and $\Gamma(T^*(M)\otimes_C E)$, respectively, the spaces of all sections of E and of the tensor product $T^*(M)\otimes_C E$. A ***connection*** is an operator ([8])

$$: \Gamma(E)\longrightarrow \Gamma(T^*(M)\otimes_C E)$$

satisfying the conditions :

i) $\nabla(\gamma_1+\gamma_2)=\nabla(\gamma_1)+\nabla(\gamma_2)$, $\forall\gamma_1, \gamma_2\in\Gamma(E)$,

ii) $\nabla(f\ )=df\cdot\gamma+f(\gamma)$, $\forall\gamma\in\Gamma(E)$,

where $f\in C^\infty(M)$ (=the space of complex-valued C^∞-functions defined on M) and $df\cdot\gamma=df\otimes_{C^\infty(M)}\gamma$

Let U be an open set of M, and let $\{e_1, \cdots, e_q\}$ be a frame field over U (*i, e.*, $\forall x\in U$, $e_1(x), \cdots\cdots, e_q(x)$ are linearly independent in E_x), and let $x^1, \cdots x^n$ be a local coordinates in U. Then we can write such that

$$\nabla e_i=\sum_j w_i^j e_j \quad (1\leqq i,\ j\leqq q) \quad (※※)$$

where $w_i^j=\sum_{k=1}^{n}f_i^{j,k}dx^k$ and $f_i^{j,k}: U\longrightarrow C$ is a C^∞-function. If we put

$$e=\begin{pmatrix} e_1\\ \vdots\\ e_q\end{pmatrix}, \qquad w=(w^j_i)=\begin{pmatrix} w_i^j\cdots w_i^q\\ \vdots \quad\ \vdots\\ w_q^1\cdots w_q^q\end{pmatrix},$$

then (※※) can be written

$$\nabla e=we\ ,$$

and we call w the ***connection matrix.*** A section $\xi\in\Gamma(E)$ is said to be ***horizontal*** if' $\nabla\xi=0$.

Definition 2.8. Under the above situation we put

$$\Omega=dw-w\wedge w$$

which is called the ***curvature matrix*** relative to the frame field e. We have to note here that

$$dw=d\begin{pmatrix} w_1^1\cdots\cdots w_1^q\\ \vdots \qquad \vdots\\ w_q^1\cdots\cdots w_q^q\end{pmatrix}=\begin{pmatrix} dw_1^1\cdots\cdots dw_1^q\\ \vdots \qquad \vdots\\ dw_q^1\cdots\cdots dw_q^q\end{pmatrix}$$

and

$$w \wedge w = \begin{pmatrix} w_1^1 \cdots\cdots w_1^q \\ \vdots \qquad \vdots \\ w_q^1 \cdots\cdots w_q^q \end{pmatrix} \wedge \begin{pmatrix} w_1^1 \cdots\cdots w_1^q \\ \vdots \qquad \vdots \\ w_q^1 \cdots\cdots w_q^q \end{pmatrix} = \left(\sum_{k=1}^{q} w_i^k \wedge w_k^j\right)$$

Proposition 2.9. $d\Omega + \Omega \wedge w - w \wedge \Omega = 0$

Proof. From $\quad \Omega = dw - w \wedge w$ we get

$$d\Omega = d^2 w - dw \wedge w + w \wedge dw$$

and $\quad$ *i.e.*, $\quad d\Omega + dw \wedge w - w \wedge dw = 0$

$$dw = \Omega + w \wedge w$$

Since $\quad dw \wedge w = (\Omega + w \wedge w) \wedge w = \Omega \wedge w + w \wedge w \wedge w$

we get $\quad d\Omega + \Omega \wedge w - w \wedge \Omega = 0$ Q. E. D.

(Note that sometimes $d\Omega + \Omega \wedge w - w \wedge \Omega = 0$ is called the *Bianchi identity*).

Definition 2.10. Let $\pi : E \longrightarrow M$ be an hermitian vector bundle (Definition 2.2) with its hermitian structure H. A connection in $\pi : E \longrightarrow M$ is said to be ***admissible*** if $H(\xi, \eta)$ is constant when ξ and η are horizontal sections along arbitrary curves.

Proposition 2.11. *Under the situation of Definition* 2.10, *let* ∇ *be a connection in* $\pi : E \longrightarrow M$ *with connection matrix* $w = (w_i^j)$. *Then* ∇ *is admissible if and only if*

$$dh_{ik} - \sum_j h_{jk} w_i^j - \sum_j h_{ij} \overline{w}_k^j = 0,$$

where $e = (e_1, \cdots\cdots\cdots, e_q)$ *is a frame field of* E ($\dim E = q$) *and*

$$h_{ij} = H(e_i, e_j) = \overline{h}_{ji} \quad (1 \leq i, j \leq q).$$

Proof. For each section $\xi \in \Gamma(E)$ we can write

$$\xi = \sum_{i=1}^{q} \xi^i e_i$$

where $\xi^i : M \longrightarrow C$ is a C^∞-function for $i = 1, \cdots, q$. Therefore, by definition of connection we have

$$\nabla \xi = \sum_{i=1}^{q} \nabla(\xi^i e_i) = \sum_{i=1}^{q} (d\xi^i \cdot e_i + \xi^i \nabla e_i) = \sum_{i=1}^{q} \left(d\xi^i + \sum_{j=1}^{q} \xi^j w_j^i\right) e_i$$

This implies that

$$\xi \text{ is } \textit{horizontal} \Longleftrightarrow \nabla \xi = 0 \Longleftrightarrow d\xi^i + \sum_{j=1}^{q} \xi^j w_j^i = 0 \quad (※※※)$$

Let us put

$$\xi = \sum_{i=1}^{q} \xi^i e_i, \quad \eta = \sum_{j=1}^{q} \eta^j e_j,$$

where $\xi^i, \eta^j : M \longrightarrow C$ are C^∞-functions.

Then

$$H(\xi, \eta) = H(\sum_{i=1}^{q} \xi^i e_i, \quad \sum_{k=1}^{q} \eta^k e_k)$$

$$= \sum_{i,k} H(e_i, e_k)\, \xi^i \bar{\eta}^k = \sum_{i,k} h_{ik}\, \xi^i \bar{\eta}^k$$

Suppose that ξ and η are horizontal. Then by (※※※) we have

$$dH(\xi, \eta) = \sum_{i,k} [(dh_{i,k})\xi^i \bar{\eta}^k + h_{ik}(d\xi^i)\bar{\eta}^k + h_{ik}\xi^i(d\bar{\eta}^k)]$$

$$= \sum_{i,k} [(dh_{ik})\xi^i \bar{\eta}^k - h_{ik}(\sum_{j=1}^{q} \xi^j w_j^i)\bar{\eta}^k - h_{ik}\xi^i(\sum_{j=1}^{q} \bar{\eta}^j \bar{w}_j^k)$$

$$= \sum_{i,k}(dh_{ik} - \sum_{j=1}^{q} h_{jk} w_i^j - \sum_{j=1}^{q} h_{ij}\bar{w}_k^j)\, \xi^i \bar{\eta}^k .$$

Hence, we have

$$dh_{ik} - \sum_j h_{jk} w_i^j - \sum_j h_{ij}\bar{w}_k^j = 0 \Longleftrightarrow H(\xi, \eta) \text{ is constant.} \qquad \text{Q. E. D.}$$

Let M be a complex manifold with $\dim_C M = n$, and let $\pi: E \longrightarrow M$ be a complex vector bundle over M with fiber dimension q. Since every vector bundle has a connection ([17]), there is a connection ∇ in the bundle $\pi: E \longrightarrow M$. If for local trivializations $\{(U, h_U), (V, h_V), \cdots\}$ all transition functions g_{UV} of E are holomorphic then the bundle E is said to be ***holomorphic***.

Let E be a holomorphic bundle. Then we have to take such that

i) each section in $\Gamma(E)$ is holomorphic,

ii) each frame field $(e_1, \cdots\cdots, e_q)$ are holomorphic

i. e., the component of $\gamma \in \Gamma(E)$ are holomorphic and e_i's are holomorphic sections. A connection such that the connection matrix is a matirx of 1-forms of type (1, 0) relative to a holomorphic frame field is called a ***connection of type*** (1, 0).

Proposition 2. 12. *Under the above situation, let H be an hermitian structure of E, and let $w = (w_i^j)$ be the connection matrix of ∇. If ∇ is an admissible connection of type* (1, 0), *then we have as its connection matrix.*

$$w = \partial H \cdot H^{-1}$$

and as its curvature matrix.

$$\Omega = -\partial\bar{\partial} H \cdot H^{-1} + \partial H \cdot H^{-1} \wedge \bar{\partial} H \cdot H^{-1}$$

Proof. We have already proved that

$$\nabla \text{ is admissible} \Longleftrightarrow dh_{ik} - \sum_j h_{jk} w_i^j - \sum_j h_{ij}\bar{w}_k^j = 0$$

(see proposition 2. 11). The last expression is denoted by

$$dH = wH + H^t \bar{w}$$

in matrix notation, where ${}^t\bar{w}$ denotes the transpose of the matrix $\bar{w}$. By proposition 2.7. we have $d = \partial + \bar{\partial}$. Hence

$$\partial H + \bar{\partial} H = wH + H\,{}^t\bar{w}$$

In both sides ∂H and wH are 1-forms of type (1, 0). It follows that $\partial H = wH$, *i.e.*, $w = \partial H \cdot H^{-1}$. Furthermore,

$$\begin{aligned} dw &= (\partial + \bar{\partial})\, w = (\partial + \partial)\,(\partial H \cdot H^{-1}) \\ &= \bar{\partial}\partial H \cdot H^{-1} - \partial H \wedge (\partial + \bar{\partial})\, H^{-1} \\ &= -\partial\bar{\partial} H \cdot H^{-1} - \partial H \wedge \partial H^{-1} - \partial H \wedge \bar{\partial} H^{-1} \\ &= -\partial\bar{\partial} H \cdot H^{-1} + \partial H \cdot H^{-1} \wedge \partial H \cdot H^{-1} + \partial H \cdot H^{-1} \wedge \bar{\partial} H \cdot H^{-1} \end{aligned}$$

because of that from

$$\partial (H \cdot H^{-1}) = \partial H \cdot H^{-1} + H \cdot \partial H^{-1} = 0$$

we get

$$\partial H^{-1} = -H^{-1}\, \partial H \cdot H^{-1}$$

Since $\Omega = dw - w \wedge w$, we have

$$\Omega = -\partial\bar{\partial} H \cdot H^{-1} + \partial H \cdot H^{-1} \wedge \partial H \cdot H^{-1} + \partial H \cdot H^{-1} \wedge \bar{\partial} H \cdot H^{-1} - \partial H \cdot H^{-1} \wedge$$
$$\partial H \cdot H^{-1} = -\partial\bar{\partial} H \cdot H^{-1} + \partial H \cdot H^{-1} \wedge \bar{\partial} H \cdot H^{-1}$$

Q. E. D.

Note that is our situation in which we call $\frac{i}{2\pi}\Omega$ the ***curvature form*** of the connection ∇.

Example 2.13. We want to illustrate an holomorphic bundle. Let P_m be the m-dimensional complex projective space. To define P_m, take $C^{m+1} - \{0\}$, where $0 = (0, \cdots, 0)$, and identify those points $(z^0, z^1, \cdots, z^m)$ $(\in C^{m+1} - \{0\})$ which differ from each other by a fact. P_m can be covered by $m+1$ open subsets U_i defined respectively by $z^i \neq 0$, $0 \leqq i \leqq m$. In U_i $(0 \leqq i \leqq m)$ we have the local coordinates ${}_i\zeta^k = z^k / z^i$, $0 \leqq k \leqq m$, $i \neq k$.

The transition of local coordinates in $U_i \cap U_j$ is given by

$${}_j\zeta^k = {}_i\zeta^h \,/\, {}_i\zeta^j \;,\quad 0 \leqq h \leqq m \quad h \neq j$$

which are holomorphic functions. There is a natural projoction

$$\psi : C^{m+1} - \{0\} \longrightarrow P_m,$$

which is a holomorphic line bundle (Note that we call it the ***universal line bundle*** over P_m). In $U_i \cap U_j$ this bundle has the transition function

$$g_{U_i U_j} = g_{ij} = {}_j\zeta^i = \frac{z^i}{z^j} \;,\qquad i \neq j,\; 0 \leqq i, j \leqq m \;.$$

For each $(z^0, \cdots, z^m) \in C^{m+1} - \{0\}$ the linear form $\sum_{i=0}^{m} a_i z^i$ (a_i : constant) has the expression

$$\sum_j a_j z^j = z^i (a_0\, {}_i\zeta^0 + \cdots + a_i + \cdots + a_m\, {}_i\zeta^m)$$

in the local coordinates in $\psi^{-1}(U_i)$ $(z^i \neq 0)$, which defines a section in the line bundle whose transition functions are

$$g'_{ij} = \frac{z^j}{z^i} = ({}_j\zeta^i)^{-1}$$

We denote this line bundle by H and call the ***hyperplane section bundle*** of P_m. It is the negative ([4]) or dual of the universal line bundle.

Definition 2.14. For a complex manifold M of rank n, let $T(M)$ be the tangent bundle of M and let H be an hermitian structure on $T(M)$. For a local coordinates $z^1, \cdots\cdots, z^n$ of M a natural frame field is given by $\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\}$. As before we put

$$h_{ik} = H\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^k}\right) = \bar{h}_{ki} \qquad (1 \leqq i, k \leqq n).$$

Then the matrix

$$H = {}^t\bar{H} = (h_{ik})$$

is positive hermitian. The *Kähler form* is defined by

$$\hat{H} = \frac{i}{2} \sum_{j,k} h_{jk}\, dz^j \wedge d\bar{z}^k,$$

which is a real-valued form of type (1, 1). An hermitian manifold is said to be *Kählerian* if the Kähler form is closed, *i.e.*,

$$d\hat{H} = 0.$$

A complex manifold is said to be *Kählerian* if it has an hermitian structure which is Kählerian.

§ 3. Some properies of $G_n(C^{n+k})$

Definition 3.1. The *complex Grassmann manifold* $G_n(C^{n+k})$ is the set of all n-dimensional planes through the origin of coordinate space.

The *complex Stiefel manifold* $V_n(C^{n+k})$ is the set of all n-frames in C^{n+k}, where an n-frame in C^{n+k} is an n-tuple of linearly independent vectors in C^{n+k}. There is a canonical projection

$$\begin{array}{cccc} q: & V_n(C^{n+k}) & \longrightarrow & G_n(C^{n+k}) \\ & \Downarrow & & \Downarrow \\ & (z^1, \cdots, z^n) & \rightsquigarrow & [z^1, \cdots, z^n] \end{array}$$

where $[z^1, \cdots, z^n]$ is the n-plane which is generated by $z^1, \cdots, z^n$. Since $V_n(C^{n+k})$ is an open set of the n-fold Cartesian product $C^{n+k} \times \cdots\cdots \times C^{n+k}$ [(14)]. We give $G_n(C^{n+k})$ the quotient topology. In fact, $G_n(C^{n+k})$ is homeomorphic to an identification space of $\dot{V}_n^0(C^{n+k})$ which is the subset of $V_n(C^{n+k})$ consisting of all orthonormal n-frames.

Proposition 3.2. *$G_n(C^{n+k})$ is a compact topological manifold* ([13]) *with dimension nk and there is a homeomorphism*

$$\begin{array}{ccc} \perp : G_n(C^{n+k}) & \longrightarrow & G_k(C^{n+k}) \\ \Cup & & \Cup \\ X & \rightsquigarrow & X^{\perp} \end{array}$$

where X is the orthogonal k-plane of X.

Proof. Step I. We shall prove that $G_n(C^{n+k})$ is Hausdorff. For a fixed point $w \in C^{n+k}$ and any $X \in G_n(C^{n+k})$, let $\rho_w(X)$ be the square of the Euclidean distance from w to X. That is, if $x_1, \cdots, x_n$ is an orthonomal basis for X then

$$\rho_w(X) = \langle w, w \rangle - \langle w, x_1 \rangle \overline{\langle w, x_1 \rangle} - \cdots - \langle w, x_n \rangle \overline{\langle w, x_n \rangle}$$

where for $w = (w^1, \cdots, w^{n+k})$ and $v = (v^1, \cdots, v^{n+k})$ in C^{n+k}

$$\langle w, v \rangle = \sum_{j=1}^{n+k} w_j \bar{v}_j$$

Therefore $\rho_w : G_n(C^{n+k}) \longrightarrow C$ is a continuous. If $X \neq Y$ in $G_n(C^{n+k})$, $w \in X$ and $w \notin Y$, then $0 = \rho_w(X) \neq \rho_w(Y)$ in C. Since C is Hausdorff, there exist open set $U(0)$ and $U(\rho_w(Y))$ such that $U(0) \cap U(\rho_w(Y)) = \phi$, where $U(0)$ is an open neighbourhood of 0 in C. and $U(\rho_w(Y))$ an open neighborhood of $\rho_w(Y)$ in C. Then open set $\rho_w^{-1}(U(0))$ and $\rho_w^{-1}(U(\rho_w(Y)))$ separate X and Y.

Step II. Since $V_n^\circ(C^{n+k})$ is compact and $q_0 : V_n^\circ(C^{n+k}) \longrightarrow G_n(C^{n+k})$ it is clear that $G_n(C^{n+k})$ is compact, where $q_0 = q \mid V_n^\circ(C^{n+k})$.

Step III. We shall prove that each point X_0 of $G_n(C^{n+k})$ has an open neighborhood U which is homeomorphic to C^{nk}. We can regard C^{n+k} as the direct sum $X_0 \oplus X_0^{\perp}$. Define

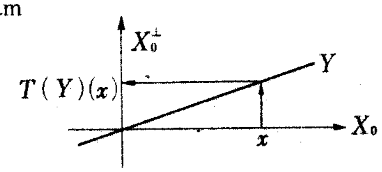
$$U = \{Y \in G_n(C^{n+k}) \mid X_0^{\perp} \cap Y = \{0\}\}$$

i.e., $Y \in U \Longleftrightarrow$ the orthogonal projection $p : X_0 \oplus X_0^{\perp} \longrightarrow X_0$ maps Y onto X_0

Then each $Y \in U$ can be considered as the graph of linear transformation,

$$T(Y) : X_0 \longrightarrow X_0^{\perp}$$

(see the diagram



).

Therefore we have an one-to-one correspondence

$$T : U \longrightarrow Hom_C(X_0, X_0^{\perp}) \cong C^{nk}$$

We have to show that T is a homeomorphism. Let $x_1, \cdots, x_n$ be a fixed orthonormal basis for X_0. Then, there exists a unique basis $y_1, \cdots, y_n$ of $Y \in U$ such that

$$p(y_1) = x_1, \cdots, p(y_n) = x_n$$

where $p : X_0 \oplus X_0^{\perp} \longrightarrow X_0$. It is obvious that the n-frame $\{y_1, \cdots, y_n\}$ depends continuously on Y. Suppose the identity

$$y_i = x_i + T(Y)\ x_i$$

(see the above diagram). Since y_i depends continuouously on Y, it follows that $T(Y)\ x_i \in X_0^{\perp}$ depends continuously on Y, That is, $T(Y)$ depends continuously on Y. Since T^{-1} is a continuous function, $G_n(C^{n+k})$ is a topological manifold with dimension nk.

Step IV. For $X \in G_n(C^{n+k})$, we shall prove that $X^{\perp} \rightsquigarrow X$ is continuous. We define a function as follows

$$f : q^{-1}U \longrightarrow V_k(C^{n+k}).$$

Let $\{\bar{x}_1, \cdots \bar{x}_k\}$ be a fixed basis for $X_0^{\perp}$. For each $(y_1, \cdots, y_n) \in q^{-1}\ U$ such that $[y_1, \cdots, y_n] = Y$, to obtain an orthonomal $(n+k)$-- frame $(y_1', \cdots, y_{n+k})$ with $y'_{n+1}, \cdots, y'_{n+k} \in Y^{\perp}$ we apply the Gram-Schmidt process to the vector $(y_1, \cdots, y_n, \bar{x}_1, \cdots, \bar{x}_k)$

Setting

$$f(y_1, \cdots\cdots, y_n) = (y'_{n+1}, \cdots\cdots, y'_{n+k})$$

we have the commutative diagram

$$\begin{array}{ccc} q^{-1}U & \xrightarrow{\ f\ } & V_k(C^{n+k}) \\ q \downarrow & & \downarrow q \\ U & \xrightarrow{\ \perp\ } & G_k(C^{n+k}) \end{array}$$

Since it is clear that f is continuous, $q \circ f$ is also continuous. This implies that $\perp$ $(Y \rightsquigarrow Y^{\perp})$ is continuous. Q. E. D.

Theorem 3. 3. $G_k(C^{n+k}) = \dfrac{GL(n+k, C)}{GL(k, n, C)} = \dfrac{U(n+k)}{U(k) \times U(n)}$

where the group $GL(k, n, C)$ *consists of all non-singular matrices of the form*

$$\underbrace{\begin{pmatrix} A & O \\ B & C \end{pmatrix}}_{k \quad\; n} \begin{matrix} \} k \\ \} n \end{matrix}$$

(the elements at the upper-right corner are zeros) and the group $U(n+k)$ *denotes the set of all* $(n+k) \times (n+k)$ *matrices.*

Proof. At first we define the map as follows

$$\eta_k^{n+k} : U(n+k) \longrightarrow V_k^0(C^{n+k}).$$

For the usual orthonormal basis $(e_1, \cdots, e_{n+k})$ of C^{n+k} and $u \in U(n+k)$ we put

$$\eta_k^{n+k}(u) = (u(e_1), \cdots, u(e_k))$$

(Note that $e_j = (0, \cdots 0, \underset{(j\text{-}th)}{1}, 0, \cdots, 0) \in C^{n+k}$), then η_k^{n+k} is continuous.

Step I. We shall prove that

$$G_k(C^{n+k}) = \frac{U(n+k)}{U(k) \times U(n)}$$

In order to prove this, we have note that for $u, v \in U(n+k)$

$$\eta_k^{n+k}(u) = \eta_k^{n+k}(v) \Longleftrightarrow u = vw, \quad \text{where } w \in I_k \times U(n),$$

and $$I_k \times U(n) = \begin{pmatrix} \begin{matrix} 1 & & \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ & \underbrace{U(n)}_{n} \end{pmatrix} \} k$$

Moreover, $(\eta_k^{n+k})^{-1}\, \eta_k^{n+k}(u)$ equals the coset $u(1_k \times U(n))$.

In fact, it is obvious that $\eta_k^{n+k}(u) = \eta_k^{n+k}(v)$ if and only if for $1 \leqq i \leqq k$ $u(e_i) = v(e_i)$ or $v^{-1} u \in I_k \times U(n)$ (Note that each element of $I_k \times U(n)$ does not change $e_1, \cdots$, and e_k). Furthermore since $\eta_k^n(u) = \eta_k^n(v) \Longleftrightarrow u = vw$ $(w \in I_k \times U(n))$ we have $(\eta_k^{n+k})^{-1} \cdot \eta_k^{n+k}(u) = u(I_k \times U(n))$. Since $\eta_k^{n+k} : U(n+k) \longrightarrow V_k^0(C^{n+k})$ is surjective, we have the homeomorphism

$$Q_k^{n+k} : U(n+k) \bmod U(n) \longrightarrow V_k^0(C^{n+k})$$

Hence we have continuous surjection

$$q_0 Q_k^{n+k} : U(n+k) \,/\, U(n) \longrightarrow G_k(C^{n+k}).$$

Next we have to note that for u mod $U(n)$ and v mod $U(n)$

$$q_0\,\theta_k^{n+k}(u)=q_0\,\theta_k^{n+k} \Longleftrightarrow v=us_1s_2,$$

where $s_1\in I_k\times U(n)$ and $s_2\in U(k)$

This is proved as follows. For $Q_k^{n+k}(u)=(u_1,\cdots,u_k)\ (u_i=u(e_i))$ and $Q_k^{n+k}(v)=(v_1,\cdots,v_k)\ \ (v_i=v(e_i))$

$$[u_1,\cdots,u_k]=[v_1,\cdots,v_k] \text{ in } G_k(C^{n+k}) \Longleftrightarrow u_i=s_2(v_i),\quad s_2\in U(k).$$

By the above description

$$u_i=u(e_i)=us_1(e_i)\ ,\qquad 1\leqq i\leqq k$$

$$s_2(v_i)=us_1(e_i)\ ,\qquad k+1\leqq i\leqq n+k$$

for $s_1\in U(n)$. Hence, from $u_i=s_2(v_i)$ we get $us_1=s_2v$ or $v=us_1s_2^{-1}$ (Note that $s_2\in U(k)\Rightarrow s_2^{-1}\in U(k)$). Therefore we have the homeomorphism

$$\begin{array}{ccc}\psi_k^{n+k}: U(n+k)\,/\,U(k)\times U(n) & \longrightarrow & G_k(C^{n+k})\\ \Downarrow & & \Downarrow\\ u \bmod U(k)\times U(n) & \rightsquigarrow & [u(e_1),\cdots,u(e_k)]\end{array}$$

Step Ⅱ. We shall prove that

$$G_k(C^{n+k})=\frac{GL(n+k,C)}{GL(k,n,C)}$$

At first we have to note that the group $GL(k,n,C)$ is the subgroup of all element of $GL(n+k,\ C)$ leaving fixed the k-dimensional subspace of C^{n+k} spanned by the first coordinate vectors. The map

$$\begin{array}{ccc}\bar{\varphi}_k^{n+k}: GL(n+k,\ C) & \longrightarrow & G_k(C^{n+k})\\ \Downarrow & & \Downarrow\\ u & \rightsquigarrow & [u(e_i),\cdots,u(e_k)]\end{array}$$

is a continuous surjection. It is clear that

$$(\bar{\varphi}_k^{n+k})^{-1}\ \ \bar{\varphi}_k^{n+k}(u)=uGL(k,n,C)$$

and thus the map

$$\begin{array}{ccc}\varphi_k^{n+k}: GL(n+k,\ C) \bmod GL(k,n,C) & \longrightarrow & G_k(C^{n+k})\\ \Downarrow & & \Downarrow\\ uGL(k,n,C) & \rightsquigarrow & [u(e_i),\cdots,u(e_k)]\end{array}$$

is a homeomorphism.

Q. E. D.

§ 4. Kählerian Structure on $G_{k+1}\ (C^{n+1})$

Each element of $G_{k+1}(C^{n+1})\ (n\geqq k)$ can be represented by a non-zero decomposable

$(k+1)$ − vector

$$A = X_0 \wedge \cdots\cdots \wedge X_k \quad (\neq 0)$$

up to a constant ($i.e.$, since $A \in G_{n+1}(C^{n+1})$ is a $(k+1)$-plane through the origin we can take $k+1$ vector $X_0, \cdots, X_k$ on A such that

i) $X_0, \cdots, X_k$ are linearly independent,

ii) A is spanned by $(X_0, \cdots, X_k)$).

Let $e_0, \cdots, e_k$ be a fixed from in C^{n+1}. Then we can denote

$$A = \sum_{\alpha} P_{\alpha_0 \cdots \alpha_k} e_{\alpha_0} \wedge \cdots \wedge e_{\alpha_k} \quad (0 \leqq \alpha_0, \cdots, \alpha_k \leqq n),$$

where since $P_{\alpha_0 \cdots \alpha_j \cdots \alpha_i \cdots \alpha_k} = -P_{\alpha_0 \cdots \alpha_i \cdots \alpha_j \cdots \alpha_k}$ P's are skew-symmetric in their indices. The $P_{\alpha_0 \cdots \alpha_k}$ are called the ***Cayley-Plücker-Grassmann coordinates*** in $G_{k+1}(C^{n+1})$. Put $\nu = {}_{n+1}C_{k+1}$ then there is an imbedding map

$$\begin{array}{ccc} G_{k+1}(C^{n+1}) & \longrightarrow & P_{\nu-1} \\ \Downarrow & & \Downarrow \\ A = \sum_{\alpha} P_{\alpha_0 \cdots \alpha_k} e_{\alpha_0} \wedge \cdots \wedge e_{\alpha_k} & \leadsto & (P_{0 \cdots k}, \cdots, P_{n-k \cdots n}), \end{array}$$

where $(P_{0 \cdots k}, \cdots, P_{n-k \cdots n})$ is a homogenous coordinate in $P_{\nu-1}$ ([4], [16]).

For $Z = (z^0, \cdots, z^n)$, $W = (w^0, \cdots, w^n) \in C^{n+1}$, we define the inner product by

$$\langle Z, W \rangle = z^0 \bar{w}^0 + \cdots + z^n \bar{w}^n = \overline{\langle W, Z \rangle} = \langle \bar{W}, \bar{Z} \rangle$$

By using these we want to define an hermitian structure in the bundle

$$\mathscr{H} : E_0 \longrightarrow G_{k+1}(C^{n+1}),$$

where

$$E_0 = \{(\nu, A) \in C^{n+1} \times G_{k+1}(C^{n+1}) \mid \nu \wedge A = 0,\ i.e.,\ \nu \in A\}.$$

For two elements

$$A = X_0 \wedge \cdots \wedge X_n, \quad M = Y \wedge \cdots \wedge Y_n \qquad (※)$$

in $G_{k+1}(C^{n+1})$, we define the ***hermitian scalar product***

$$\langle A, M \rangle = \det(\langle X_\alpha, Y_\beta \rangle), \quad (0 \leqq \alpha, \beta \leqq k)$$

then $\langle A, M \rangle$ depends only on A and M and which is independent of the ways that they are decomposed in (※).

We will introduce the notations

$$| A, M | = | \langle A, M \rangle |, \quad |A| = \langle A, A \rangle^{\frac{1}{2}},$$

where $|A|$ is called the ***norm*** of A. Then we have the Schwartz inequality

$$|A, M| \leqq |A| |M|.$$

Definition 4.1. An ***unitary*** $(h+1)$ ***frame*** is an orderd set of $h+1$ vectors $Z_0, \cdots, Z_h$

satisfying

$$\langle Z_i, Z_j \rangle = \delta_{ij}\ ,\quad 0 \leqq i, j \leqq h$$

If $h=n$, then $Z_0, \cdots, Z_n$ is called a unitary frame. By this definition we easily see that $U(n+1)$ is the set of all unitary frames (Note that if $Z_0 = (z_0^0, \cdots, z_0^n), \cdots, Z_n = (z_n^0, \cdots, z_n^n)$ is a unitary frame, then

$$\begin{pmatrix} z_0^0 & \cdots\cdots & z_0^n \\ \vdots & & \vdots \\ z_n^0 & \cdots\cdots & z_n^n \end{pmatrix} \in U(n+1) \quad).$$

In our situation we have the fiber-rings ([4], [7]),

$$U(n+1) \xrightarrow{\ \lambda\ } V_{k+1}^0(C^{n+1}) \xrightarrow{\ \mu\ } G_{k+1}(C^{n+1})\ , \qquad (※※)$$

where for $(Z_0, \cdots, Z_n) \in U(n+1)$

$$\lambda(Z_0, \cdots, Z_n) = (Z_0, \cdots, Z_k) \in V_{k+1}^0(C^{n+1})$$

$$\mu(Z_0, \cdots, Z_k) = Z_0 \wedge \cdots \wedge Z_k \in G_{k+1}(C^{n+1}).$$

For $(Z_0, \cdots, Z_n) \in U(n+1)$, we define

$$\theta_{\alpha\beta} = \langle dZ_\alpha,\ Z_\beta \rangle\ , \qquad 0 \leqq \alpha, \beta \leqq n$$

Proposition 4.2. *Under the above definition the following hold.*

i) $\theta_{\alpha\beta} + \overline{\theta}_{\beta\alpha} = 0$

ii) $dZ_\alpha = \sum_\beta \theta_{\alpha\beta} Z_\beta$

iii) $d\theta_{\alpha\beta} = \sum_\gamma \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta}\ , \qquad 0 \leqq \alpha, \beta, \gamma \leqq n$

Proof. i) Since $\langle Z_\alpha, Z_\beta \rangle = \delta_{\alpha\beta}$ we have

$$d\langle Z_\alpha, Z_\beta \rangle = \langle dZ_\alpha, Z_\beta \rangle + \langle d\overline{Z}_\beta, \overline{Z}_\alpha \rangle$$

$$= \langle dZ_\alpha, Z_\beta \rangle + \overline{\langle dZ_\beta, Z_\alpha \rangle} = 0$$

and thus we have $\theta_{\alpha\beta} + \overline{\theta}_{\beta\alpha} = 0$.

ii) For $(Z_0, Z_1, \cdots, Z_n) \in U(n+1)$ we put

$$Z_0 = (z_0^0, \cdots, z_0^n)$$
$$\cdots\cdots\cdots$$
$$Z_n = (z_n^0, \cdots, z_n^n).$$

Then

$$\begin{pmatrix} z_0^0 & z_0^1 & \cdots\cdots & z_0^n \\ \vdots & \vdots & & \vdots \\ z_n^0 & z_n^1 & \cdots\cdots & z_n^n \end{pmatrix}$$

is an unitary matrix.

So we have

$$\sum_k z_k^i \bar{z}_k^j = \sum_k z_i^k \bar{z}_j^k = \delta_{ij}$$

From $\theta_{\alpha\beta} = \langle dZ_\alpha, Z_\beta \rangle$ we have

$$\sum_\beta \theta_{\alpha\beta} Z_\beta = \sum_\beta \langle dZ_\alpha, Z_\beta \rangle Z_\beta = (dz_\alpha^0 z_0^0 + \cdots + dz_\alpha^n \bar{z}_0^n) \begin{pmatrix} z_0^0 \\ \vdots \\ z_0^n \end{pmatrix} + \cdots$$

$$+ (dz_\alpha^0 \bar{z}_n^0 + \cdots + dz_\alpha^n \bar{z}_n^n) \begin{pmatrix} z_n^0 \\ \vdots \\ z_n^n \end{pmatrix}$$

$$= \begin{pmatrix} dz_\alpha^0 (z_0^0 \bar{z}_0^0 + \cdots + z_n^0 \bar{z}_n^0) + \cdots + dz_\alpha^n (\bar{z}_0^n z_0^0 + \cdots + \bar{z}_n^n z_n^0) \\ \vdots \\ dz_\alpha^0 (z_0^n \bar{z}_0^0 + \cdots + z_n^n \bar{z}_n^0) + \cdots + dz_\alpha^n (z_0^n \bar{z}_0^n + \cdots + z_n^n \bar{z}_n^n) \end{pmatrix}$$

$$= \begin{pmatrix} dz_\alpha^0 \\ \vdots \\ dz_\alpha^n \end{pmatrix} = dZ_\alpha$$

iii) Since $d^2 = 0$, from (ii) we have

$$\sum_\beta d\theta_{\alpha\beta} Z_\beta - \sum_\beta \theta_{\alpha\beta} \wedge dZ_\beta = 0,$$

and thus

$$\sum_\beta d\theta_{\alpha\beta} Z_\beta = \sum_\beta \theta_{\alpha\beta} \wedge dZ_\beta$$

$$= \sum_\beta (\theta_{\alpha\beta} \wedge \sum_\gamma \theta_{\beta\gamma} Z_\gamma)$$

$$= \sum_\beta (\sum_\gamma \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta}) Z_\beta \quad .$$

This implies that $d\theta_{\alpha\beta} = \sum_\gamma \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta}$. Q. E. D.

Under the projection $\mu \circ \lambda$ ((※※)), a form w on $G_{k+1}(C^{n+1})$ is completely determined by the image $(\mu \circ \lambda)^* w$ ([4]). In order to study about forms on $G_{k+1}(C^{n+1})$, we may study about forms on $U(n+1)$.

Let Λ be an element of $G_{k+1}(C^{n+1})$ such that

$$\Lambda = X_0 \wedge \cdots \wedge X_k.$$

We put

$$\Lambda_0 = \frac{\Lambda}{|\Lambda|} = Z_0 \wedge \cdots \wedge Z_k,$$

then $Z_0, \cdots, Z_k$ is an unitary $(k+1)$ — frame.

Lemma 4.3. *The following hold.*

i) $\langle d\Lambda_0, \Lambda_0 \rangle = \sum_\alpha \theta_{\alpha\alpha} = -\sum_\alpha \bar{\theta}_{\alpha\alpha}$,

ii) $\langle d\Lambda_0, d\Lambda_0 \rangle = [\sum_\alpha \theta_{\alpha\alpha}][\sum_\alpha \bar{\theta}_{\alpha\alpha}] + \sum_{\alpha\gamma} \theta_{\alpha\gamma} \bar{\theta}_{\alpha\gamma}$,

where $0 \leq \alpha \leq k$, $k+1 \leq \gamma \leq n$ *and* $d\Lambda_0$ *in* ii) *is* $(\mu \circ \lambda)^* (d\Lambda_0)$. *Furthermore, the multiplication of differential forms in* ii) *is in the sense of ordinary commutative multiplication.*

Proof. Since

$$dA_0 = dZ_0\wedge Z_1\wedge\cdots\wedge Z_k+\cdots+Z_0\wedge\cdots\wedge dZ_k$$

$$\langle dA_0, A_0\rangle = \begin{vmatrix} \langle dZ_0, Z_0\rangle & \cdots & \langle dZ_0, Z_k\rangle \\ 0 & 1 \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots 0 & 1 \end{vmatrix} + \cdots + \begin{vmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 \cdots & 1 & 0 \\ \langle dZ_n, Z_0\rangle & \cdots & \langle dZ_n, Z_n\rangle \end{vmatrix}$$

$$= \begin{vmatrix} \theta_{00} & \cdots & \theta_{0k} \\ 0 & 1\cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{vmatrix} + \cdots + \begin{vmatrix} 1 & \cdots 0 & 0 \\ & \ddots & \vdots \\ 0 & 1 & \vdots \\ \theta_{k0} & \cdots & \theta_{kk} \end{vmatrix}$$

$$= \sum_\alpha \theta_{\alpha\alpha} = -\sum_\alpha \bar{\theta}_{\alpha\alpha} \quad \text{(by (i) of proposition 4.2.)}.$$

ii) Let $(Z_0, \cdots, Z_k, \cdots, Z_n)\in U(n+1)$ be an element in $(\mu\circ\lambda)^{-1}A_0$. We also put $A_0 = (Z_0, \cdots, Z_n)$,

Then we may regard $Z_{k+1}, \cdots, Z_n$ as constants. Then we have the following :

$$\langle dA_0, dA_0\rangle = \langle dZ_0\wedge Z_1\wedge\cdots\wedge Z_n+\cdots+Z_0\wedge\cdots\wedge dZ_k\wedge dZ_{k+1}\wedge\cdots\wedge Z_n,$$
$$dZ_0 \wedge\cdots\wedge Z_k+\cdots Z_0\wedge\cdots\wedge dZ_k\wedge Z_{k+1}\wedge\cdots\wedge Z_n\rangle$$

$$= \begin{vmatrix} \langle dZ_0, dZ_0\rangle & \cdots\cdots & \langle dZ_0, Z_n\rangle \\ \langle Z_1, dZ_0\rangle & 1\ 0 \cdots & 0 \\ \vdots & 0 \ddots & \vdots \\ & & 0 \\ \langle Z_n, dZ_0\rangle & 0 \cdots 0 & 1 \end{vmatrix} + \begin{vmatrix} \langle dZ_0, Z_0\rangle & \langle dZ_0, dZ_1\rangle & \cdots\cdots & \langle dZ_0, Z_n\rangle \\ 0 & \langle Z_1, dZ_1\rangle & 0\cdots & 0 \\ \vdots & \vdots & 1\ 0 \cdots & 0 \\ & & 0 \ddots & 0 \\ 0 & \langle Z_n, dZ_1\rangle & 0 \cdots 0 & 1 \end{vmatrix} + \cdots\cdots$$

$$\cdots + \begin{vmatrix} \langle dZ_0, Z_0\rangle & \cdots\cdots & \langle dZ_0, dZ_k\rangle & \cdots & \langle dZ_0, Z_n\rangle \\ 0 & 1\ 0\cdots 0 & \langle Z_1, dZ_k\rangle & 0 \cdots & 0 \\ \vdots & 0\ 1\ 0 \cdots 1 & \vdots & 0 \cdots & 0 \\ \vdots & \vdots\ 0 \quad 0 & \langle Z_k, dZ_k\rangle & 0 \cdots & 0 \\ & & & 1 \ddots & \vdots \\ 0 & 0 \cdots 0 & \langle Z_n, dZ_k\rangle & 0 \cdots & 1 \end{vmatrix} + \begin{vmatrix} \langle Z_0, dZ_0\rangle & 0 & \cdots\cdots & 0 \\ \langle dZ_1, dZ_0\rangle & \langle dZ_1, Z_1\rangle & \cdots\cdots & \langle dZ_1, Z_n\rangle \\ \vdots & 0 & 1\ 0 \cdots & 0 \\ & \vdots & 0 \ddots & 0 \\ \langle Z_n, dZ_0\rangle & 0 & 0 \cdots 0 & 1 \end{vmatrix} + \cdots$$

$$\cdots + \begin{vmatrix} 1\ 0 \cdots 0 & \langle Z_0, dZ_k\rangle & 0 \cdots 0 \\ 0 \ddots 0 & \vdots & \vdots \\ 0 \cdots 0\ 1 & \vdots & 0 \cdots 0 \\ \langle dZ_k, Z_0\rangle \cdots & \langle dZ_k, dZ_k\rangle & \cdots \langle dZ_k, Z_n\rangle \\ 0 \cdots 0 & \vdots & 1 \ddots 0 \\ 0 \cdots 0 & \langle Z_n, dZ_k\rangle & 0 \cdots 0\ 1 \end{vmatrix}$$

$$= \langle dZ_0, dZ_0\rangle + \langle dZ_0, Z_0\rangle\langle Z_1, dZ_1\rangle+\cdots+\langle dZ_0, Z_0\rangle\langle Z_k, dZ_k\rangle+\cdots\cdots+$$
$$\langle dZ_1, Z_1\rangle\langle Z_k, dZ_k\rangle+\cdots+\langle dZ_k, dZ_k\rangle = \left(\sum_\alpha \theta_{\alpha\alpha}\right)\left(\sum_\alpha \bar{\theta}_{\alpha\alpha}\right)+\sum_{\alpha\gamma}\theta_{\alpha\gamma}\bar{\theta}_{\alpha\gamma}$$

because of that

i) $\langle dZ_\alpha, dZ_\alpha\rangle = \langle \sum_\beta \theta_{\alpha\beta}Z_\beta, \ \sum_\beta \theta_{\alpha\beta}Z_\beta\rangle$

$= \sum_{\beta} \theta_{\alpha\beta} \cdot \overline{\theta}_{\alpha\beta}$. $\quad (0 \leq \beta \leq n,\ 0 \leq \alpha \leq k)$

ii) $\theta_{\alpha\beta}\overline{\theta}_{\alpha\beta} + \theta_{\beta\alpha}\overline{\theta}_{\beta\alpha} = 0$, $\quad (0 \leq \alpha,\ \beta \leq k)$

iii) $\langle dZ_\alpha, Z_\alpha \rangle \langle Z_\beta, dZ_\beta \rangle = \theta_{\alpha\alpha}\overline{\theta}_{\beta\beta}$, $\quad (0 \leq \alpha,\ \beta \leq k)$

Q. E. D.

Corollary 4.4. *We have the following* :

$$\langle dA_0, dA_0 \rangle - \langle dA_0, A_0 \rangle \langle A_0, dA_0 \rangle = \sum_{\alpha,\gamma} \theta_{\alpha\gamma}\overline{\theta}_{\alpha\gamma},$$

where $0 \leq \alpha \leq k$ *and* $k+1 \leq \gamma \leq n$

Proof. By i) of Lemma 4.3.

$$\langle dA_0, A_0 \rangle = \sum_{\alpha} \theta_{\alpha\alpha} \quad , \qquad \langle A_0, dA_0 \rangle = \sum_{\alpha} \overline{\theta}_{\alpha\alpha}$$

we have

$$\langle dA_0, A_0 \rangle \langle A_0, dA_0 \rangle = (\sum_{\alpha} \theta_{\alpha\alpha})(\sum_{\beta} \overline{\theta}_{\beta\beta}).$$

Therefore, it follows from ii) of Lemma 4.3. that

$$\langle dA_0, dA_0 \rangle - \langle dA_0, A_0 \rangle \langle A_0, dA_0 \rangle = \sum_{\alpha,\gamma} \theta_{\alpha\gamma}\overline{\theta}_{\alpha\gamma},$$

where $0 \leq \alpha \leq k$ and $k+1 \leq \gamma \leq n$.

Q. E. D.

Lemma 4.5. *If we put*

$$A_0 = A / |A| ,$$

then we have the following

$$\frac{1}{|A|^4} \{ \langle dA, dA \rangle \langle A, A \rangle - \langle dA, A \rangle \langle A, dA \rangle \} = \sum_{\alpha,\gamma} \theta_{\alpha\gamma}\overline{\theta}_{\alpha\gamma} \quad ,$$

where $0 \leq \alpha \leq k$ and $k+1 \leq \gamma \leq n$.

Proof. Form $\frac{A}{|A|}$ we get

$$dA_0 = \frac{dA}{|A|} - \frac{A}{|A|^2} d|A|, \quad d|A| = \frac{1}{2|A|} (\langle dA, A \rangle + \langle d\overline{A}, \overline{A} \rangle) \qquad (※※※$$

Let us put

$$\overline{d|A|} = \text{the 1-form with conjugate coefficient of } d|A| .$$

Then we have the following.

$$\langle dA_0, dA_0 \rangle - \langle dA_0, A_0 \rangle \langle A_0, dA_0 \rangle = \left\langle \frac{dA}{|A|} - \frac{A}{|A|^2} d|A|,\ \frac{dA}{|A|} - \frac{A}{|A|^2} d|A| \right\rangle$$
$$- \left\langle \frac{dA}{|A|} - \frac{A}{|A|^2} d|A|,\ \frac{A}{|A|} \right\rangle \left\langle \frac{A}{|A|},\ \frac{dA}{|A|} - \frac{A}{|A|^2} d|A| \right\rangle$$

$$= \frac{1}{|A|^2} \langle dA, dA \rangle - \frac{\overline{d|A|}}{|A|^3} \langle dA, A \rangle - \frac{d|A|}{|A|^3} \langle A, dA \rangle + \frac{d|A| \cdot \overline{d|A|}}{|A|^4} \langle A, A \rangle$$
$$- \left\{ \frac{1}{|A|^2} \langle dA, A \rangle - \frac{d|A|}{|A|^3} \langle A, A \rangle \right\} \left\{ \frac{1}{|A|^2} \langle A, dA \rangle - \frac{\overline{d|A|}}{|A|^3} \langle A, A \rangle \right\}$$

$$=\frac{1}{|A|^4}\langle dA, dA\rangle\langle A, A\rangle-\frac{1}{|A|^4}\langle dA, A\rangle\langle A, dA\rangle-\frac{\overline{d|A|}}{|A|^3}\langle dA, A\rangle$$

$$+\frac{\overline{d|A|}}{|A|^5}\langle dA, A\rangle\langle A, A\rangle-\frac{d|A|}{|A|^3}\langle A, dA\rangle+\frac{d|A|}{|A|^5}\langle A, A\rangle\langle A, dA\rangle$$

$$+\frac{d|A|\overline{d|A|}}{|A|^4}\langle A, A\rangle-\frac{d|A|\overline{d|A|}}{|A|^6}\langle A, A\rangle^2$$

$$=\frac{1}{|A|^4}\{\langle dA, dA\rangle\langle A, A\rangle-\langle dA, A\rangle\langle A, dA\rangle\}$$

Therefore, from Corollary 4.4 our result is obtained. Q. E. D.

Definition 4.6. For $A=X_0\wedge\cdots\wedge X_k\in G_{k+1}(C^{n+1})$ with

$$A_0=A/|A|=Z_0\wedge\cdots\wedge Z_k,$$

we define $\sum_{i,j}H(X_i,X_j)\,dZ_i\wedge d\overline{Z}_j=\langle dA, dA\rangle\langle A, A\rangle-\langle dA, A\rangle\langle A, dA\rangle$

$$=|A|^4\sum_{\alpha,\gamma}\theta_{\alpha\gamma}\overline{\theta}_{\alpha\gamma}\quad,\qquad (※※※※)$$

where $0\leq i,j,\alpha\leq k$ and $k+1\leq\gamma\leq n$. Then

$$\sum_{i,j}H(X_i,X_j)\,dZ_i\wedge d\overline{Z}=\langle dA, dA\rangle\langle A, A\rangle-\langle A, dA\rangle\langle dA, A\rangle$$
$$=\overline{\langle dA, dA\rangle\langle A, A\rangle-\langle dA, A\rangle\langle A, dA\rangle}$$

which means that

$$H(X_i,X_j)=\overline{H(X_j,X_i)}.$$

Therefore H is an *hermitian structure* in $G_{k+1}(C^{n+1})$. Also, the left side of (※※※※) implies that H is positive definite,

Theorem 4.7. i) *The complex Grassmann manifold is Kählerian.* (See Definition 2.14)

ii) *As the statements in the first part of this section $G_{k+1}(C^{n+1})$ is imbedded in P_ν, where $\nu={}_{n+1}C_{k+1}-1$. Let the curvature form of the hyperplane section bundle* (see Example 2.13) *over $G_{k+1}(C^{n+1})\subset P_\nu$ defined by the norm $|A|$ denote by A. Then*

$$\pi A=\textit{the Kähler form } \hat{H}_k \textit{ of } G_{k+1}(C^{n+1}).$$

Proof. i) From Definition 4.6., we have

$$\hat{H}_k=\frac{i}{2}\sum_{\alpha,\gamma}\theta_{\alpha\gamma}\wedge\overline{\theta}_{\alpha\gamma}\qquad\text{(See Definition 2.14)}$$

$$=\frac{1}{2i}d\,(\sum_\alpha\theta_{\alpha\alpha})\qquad\text{(See iii) of Proposition 4.2)}$$

Hence

$$d\hat{H}_k=\frac{1}{2i}d^2\,(\sum_\alpha\theta_{\alpha\alpha})=0$$

and thus $G_{k+1}(C^{n+1})$ is Kählerian.

ii) At first we shall prove that

$$\langle dA_0, A_0\rangle = \sum_{\alpha} \theta_{\alpha\alpha} = (\partial - \bar{\partial}) \log |A|.$$

From (※※※),

$$\langle dA_0, A_0\rangle = \langle \frac{dA}{|A|} - \frac{A}{|A|^2} d|A|, \frac{A}{|A|}\rangle$$

$$= \frac{1}{|A|^2} \langle dA, A\rangle - \frac{d|A|}{|A|^3} \langle A, A\rangle = \frac{1}{|A|^2} \langle dA, A\rangle - \frac{d|A|}{|A|}$$

$$= \frac{1}{2|A|^2} (\langle dA, A\rangle - \langle d\bar{A}, \bar{A}\rangle) = \frac{1}{2|A|^2} (\langle \partial A, A\rangle - \langle \bar{\partial}\bar{A}, \bar{A}\rangle).$$

On the other hand

$$(\partial - \bar{\partial}) \log |A| = (\partial - \bar{\partial}) \log \langle A, A\rangle^{\frac{1}{2}}$$

$$= \frac{1}{2} (\partial - \bar{\partial}) \log \langle A, A\rangle$$

$$= \frac{1}{2|A|^2} (\partial\langle A, A\rangle - \bar{\partial}\langle A, A\rangle)$$

$$= \frac{1}{2|A|^2} (\langle \partial A, A\rangle - \langle \bar{\partial}\bar{A}, \bar{A}\rangle)$$

Hence from i), we have

$$\hat{H}_k = \frac{1}{2i} d((\partial - \bar{\partial}) \log |A|)$$

As Example 2. 13)

$$C^{\nu} - \{0\} \longrightarrow P_{\nu-1}$$

defines the universal line bundle over P_ν and an hermitian structure is introduced in this bundle by the norm $|A|$. The restriction of this bundle to $G_{k+1}(C^{n+1}) \subset P_\nu$ is the negative of the hyperplane section bundle means in ii) and $|A|^{-1}$ define an hermitian structure on it (See Example 2. 13).

In proposition 2. 12 if $q=1$

$$H = (h) \quad , \quad \Omega = (\Omega)$$

and

$$\Omega = -\partial\bar{\partial}H \cdot H^{-1} + \partial H \cdot H^{-1} \wedge \bar{\partial}H \cdot H^{-1}$$

can be written

$$\Omega = -\partial\bar{\partial} \log h,$$

because of that

$$-\partial\bar{\partial} \log h = -\partial(\frac{\bar{\partial}h}{h}) = -\frac{1}{h^2} (\partial\bar{\partial}h \cdot h - \partial h \wedge \bar{\partial}h)$$

$$= -\partial\bar{\partial} h\cdot h^{-1} + \partial h\cdot h^{-1} \wedge \bar{\partial} h\cdot h^{-1}$$

Hence we have

$$A = \frac{i}{2\pi}\Omega = \frac{1}{2\pi i}\partial\bar{\partial}\log h,$$

where $h = |\Lambda|^{-2}$ is the square of the norm of local holomorphic section (Example 2.13). There are

$$A = \frac{1}{2\pi i}\partial\bar{\partial}\log|\Lambda|^{-2}$$

$$= -\frac{1}{\pi i}\partial\bar{\partial}\log|\Lambda|$$

$$= \frac{i}{\pi}\partial\bar{\partial}\log|\Lambda| = \frac{1}{\pi}\hat{H}_k \qquad \text{Q. E. D.}$$

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