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G-Bundles and Representation Rings

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§1. Introduction

There are a number of recently developed theories in topology such as K-theories, cobordism theory and index theorem [4,6,7,10,15]. Their influences are quite significant. On the other hand, the theory of representation rings is very useful in developing K-theories and cobordism theory.

In this paper, our main object is to prove Theorem 2.5 on the necessary and sufficient condition that a principal bundle have a restriction, and Theorem 5.4 on certain relations between representation rings and K-rings.

In Section 2, we discuss some relations between principal G-bundles and principal H-bundles, where H is a closed subgroup of a topological group G, and, by using them, we prove Theorem 2.5 which deals with the necessary and sufficient condition that every numerable $\dot{G}$-bundle restrict to a principal H-bundle.

Sections 3 and 4 deals with preparations for Sections 5. In Section 3, we give certain properties of G-moduls, which are necessary in the sequel, as Propositions 3.3 and 3.6 and Lemmata 3.7 and 3.8. In Section 4, we also give some necessary properties of representation rings as Propositions 4.3

and 4.5.

In Section 5, we first define vector bundles with G-structure by η in Definition 5.1, and then define λ-rings $K_F^G\eta(X)$ in Definition 5.2 and Proposition 5.3. Finally we prove Theorem 5.4 which deals with relations between the representation ring $R_F(G)$ of G and $K_F^G\eta(X)$.

§ 2. G-bundles

Throughout this paper, G always denotes a topological group. Let $\xi=(E(\xi), \pi, B(\xi))$ be a (real or complex) principal G-bundle, and let F be a left G-space. For $(x,y) \in E(\xi) \times F$ and $s \in G$ we define

$$(x,y)s=(xs, s^{-1}y)$$

then $E(\xi) \times F$ is a right G-space. We put

$$E(\xi)_F = (E(\xi)\times F) \bmod G$$

and define $\pi_F : E(\xi)_F \to B(\xi)$ by the commutative diagram

$$\begin{array}{ccccc} E(\xi)\times F & \xrightarrow{p_1} & E(\xi) & \xrightarrow{\pi} & B(\xi) \\ & {\scriptstyle\text{proj}}\searrow & & \nearrow{\scriptstyle\pi_F} & \\ & & E(\xi)_F & & \end{array}$$

where proj is the projection and $p_1(x,f)=x$ for $(x,f) \in E(\xi) \times F$. Then $(E(\xi)_F, \pi_F, B(\xi))$ is a fibre bundle with fibre F. In particular, if ξ is locally trivial, then so is $(E(\xi)_F, \pi_F, B(\xi))$.

Definition 2.1 With the above notations, the fibre bundle $(E(\xi)_F, \pi_F, B(\xi))$, denoted $\xi[F]$, is called the *associated fibre bundle* of ξ.

It is important to note that $\xi[G] \cong \xi$.

For each space B, let $K_G(B)$ denote the set of all isomorphism classes of all (real or complex) numerable principal G-bundles over B (for the word "*numerable*", see [5]).

Definition 2.2 A numerable principal G-bundle $\omega = (E_0, \pi_0, B_0)$ is *universal* if

$$\phi_\omega : [\quad , B_0] \to K_G$$

is an isomorphism, where $\phi_\omega(f) = f^*\omega$ for $f \in [B, B_0]$ and $[B, B_0]$ is the set of homotopy classes of continuous maps from B to B_0.

By Milnor ([8],[9]) every topological group G always has a unique universal principal G-bundle $\omega_G = (E(\omega_G), \pi_G, B_G)$ up to G-bundle isomorphism.

Throughout this paper, H always denotes a closed subgroup of G.

Definition 2.3 Let $\xi = (X, p, B)$ be a principal G-bundle and $\eta = (Y, q, B)$ a principal H-bundle. If $f : Y \to f(Y) \subset X$ is a homeomorphism onto the closed subset $f(Y)$ such that

$$f(ys) = f(y)s$$

for $y \in Y$ and $s \in G$, then η is called *restriction* of ξ, and ξ a *prolongation* of η.

Note that every principal H-bundle ($H \hookrightarrow G$) always has its prolongation. But, there does not always exist a restriction of $\xi = (X, p, B)$.

The following facts are known:

(i) A principal G-bundle ξ has a restriction to a principal H-bundle

$\eta = (Y, q, B)$ if and only if $\xi[G \bmod H]$ has a cross section.

(ii) Let $\omega_H = (E(\omega_H), \pi_H, B_H)$ be a universal principal H-bundle and $\omega_G = (E(\omega_G), \pi_G, B_G)$ a universal principal G-bundle. For a numerable principal G-bundle $\xi = (X, p, B)$ with classifying map $f: B \to B_G$, the restrictions $\eta = (Y, q, B)$ of ξ are in bijective correspondence with homotopy classes of maps $g: B \to B_H$ such that $f_0 \circ g \simeq f$ as the homotopic commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\quad f \quad} & B_G \\ & g \searrow \quad \ominus \quad \nearrow f_0 & \\ & B_H & \end{array}$$

where $(h_0, f_0) : \omega_H[G] \to \omega_G$, that is,

$$\begin{array}{ccc} E(\omega_H[G]) & \xrightarrow{\quad h_0 \quad} & E(\omega_G) \\ {\scriptstyle (\pi_H)_G} \downarrow & \copyright & \downarrow {\scriptstyle \pi_G} \\ B_H & \xrightarrow[\quad f_0 \quad]{} & B_G \end{array}$$

Proposition 2.4 Let e_G be the identity of G. A principal G-bundle $\xi = (X, p, B)$ has a restriction to the subgroup $\{e_G\}$ if and only if ξ is trivial.

Proof. It is easy to prove that if $\eta = (Y, q, B)$ is a principal $\{e_G\}$-bundle, then η is trivial. Since η has a prolongation which is trivial because η is trivial, $\xi = (X, p, B)$ is trivial. This shows the necessity.

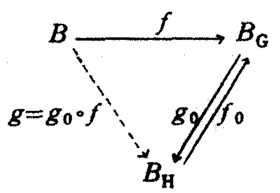
Conversely, assume that $\xi = (X, p, B)$ is trivial. Then there is a bundle isomorphism $\varphi: X \cong B \times G$. Since G mod$\{e_G\} = G$, we have $\xi[G \bmod \{e_G\}] = \xi$. So, there is a cross section

$$\sigma: B \to X$$

such that for all $b \in B$, $\sigma(b) = \varphi^{-1}(b, e)$ where e is a fixed element of G. By (i) ξ has a restriction $\eta = (Y, q, B)$ which is a principal $\{e_G\}$-bundle. $q.e.d.$

Theorem 2.5. Every numerable principal G-bundle $\xi = (X, p, B)$ has a restriction to a principal H-bundle if and only if the map $f_0: B_H \to B_G$ has a left homotopy inverse $g_0: B_G \to B_H$. If there exists g_0, then it is unique up to homotopy. If H is a deformation retract of G, then every numerable principal G-bundle restricts to a principal H-bundle.

Proof. If there exists a map $g_0: B_G \to B_H$ such that $g_0 \circ f_0 \simeq 1_{B_H}$, then we have the map $g = g_0 \circ f: B \to B_H$ as in the diagram



Then, $f_0 \circ g = f_0 \circ g_0 \circ f \simeq f$, because $f_0 \circ g_0 \simeq 1_{B_G}$. By (ii) every numerable principal G-bundle $\xi = (X, p, B)$ restricts to a principal H-bundle $\eta = (Y, q, B)$.

Conversely, we assume that every numerable principal G-bundle restricts to a principal H-bundle. Then a unversal principal G-bundle

$\omega_G = (E(\omega_G), \pi_G, B_G)$ has its restriction $\eta = (Y, q, B_G)$. Thus, by (ii) there exists a map $g_0 : B_G \to B_H$ such that $f_0 \circ g_0 \simeq 1_{B_G}$ and $g_0^* \omega_H \cong \eta$.

$$\begin{array}{ccc} B_G & \xrightarrow{\ 1_{B_G}\ } & B_G \\ & {\scriptstyle g_0} \searrow \quad \nearrow {\scriptstyle f_0} & \\ & B_H & \end{array}$$

Since $f_0^* \omega_G = \omega_H[G]$ and $g_0^*(\omega_H[G]) = \eta[G] = \omega_G$, we have

$$f_0^*(g_0^*(\omega_H[G])) = \omega_H[G].$$

This means that $g_0 \cdot f_0 \simeq 1_{B_H}$.

We suppose that there are two maps $g_0^1, g_0^2 : B_G \to B_H$ such that $f_0 \circ g_0^1 \simeq 1_{B_G} \simeq f_0 \circ g_0^2$ and $g_0^1 \cdot f_0 \simeq 1_{B_H} \simeq g_0^2 \circ f_0$. From the expression $f_0 \cdot g_0^1 \simeq 1_{B_G} \simeq f_0 \circ g_0^2$, since $f_0 \simeq f_0$ we have $g_0^1 \simeq g_0^2$ ([12]).

Finally, we assume that H is a deformation retract. Then, there is a retraction $r : G \to H$ such that $r \circ i \simeq 1_H$ and $i \circ r \simeq 1_G$ where $i : H \to G$ is the inclusion map. By the Milnor construction there is the natural inclusion $\tilde{i}_0 : E(\omega_H) \to E(\omega_G)$, and there is the natural map $\tilde{r}_0 : E(\omega_G) \to E(\omega_H)$. Then, by the commutative diagram

$$\begin{array}{ccc} E(\omega_H) & \xrightarrow{\ \tilde{i}_0\ } & E(\omega_G) \\ {\scriptstyle \pi_H} \downarrow & \copyright & \downarrow {\scriptstyle \pi_G} \\ B_H & \xrightarrow{\ i_0\ } & B_G \end{array} \quad \text{and} \quad \begin{array}{ccc} E(\omega_G) & \xrightarrow{\ \tilde{r}_0\ } & E(\omega_H) \\ {\scriptstyle \pi_G} \downarrow & \copyright & \downarrow {\scriptstyle \pi_H} \\ B_G & \xrightarrow{\ r_0\ } & B_H \end{array},$$

we have bundle maps $(\tilde{i}_0, i_0)$ and $(\tilde{r}_0, r_0)$. Since $i_0^* \omega_G = \omega_H[G]$ and $r_0 \circ i_0 \simeq 1_{B_H}$, by the first part of our theorem every numerable principal G-bundle has a restriction to a principal H-bundle, *q.e.d.*

§3. G-modules

In the sequel, F denotes either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$.

Definition 3.1 An F-vector space M is a *left G-module* if there exists a map $u: G\times M\to M$ where $u(s,m)=sm$ for $s\in G$ and $m\in M$, satisfying the following conditions:

(i) $s(m+n)=sm+sn$ for $s\in G$ and $m,n\in M$.

(ii) $(ss')(m)=s(s'm)$ for $s,s'\in G$ and $m\in M$.

(iii) $e_G m=m$, where e_G is the identity of G and $m\in M$.

(iv) $s(\alpha m)=\alpha(sm)$ for $s\in G$ and $m\in M$, $\alpha\in F$.

In the sequel, by a G-module we mean a left G-module over F.

Definition 3.2 Let M and N be G-modules. A function $f: M\to N$ is called a *G-morphism* if it satisfies the following:

(i) f is an F-linear map.

(ii) For all $s\in G$ and $m\in M$, $f(sm)=sf(m)$.

Let $\mathrm{Hom}_G(M,N)$ denote the set of G-morphisms $f: M\to N$. Then we have

$$\mathrm{Hom}_G(M,N)\subset \mathrm{Hom}_F(M,N).$$

We again assume that M and N are G-modules. Then, $M\oplus N$, $M\otimes_F N$, $\Lambda^r M$ and $\mathrm{Hom}_F(M,N)$ become G-modules by the following ways.

(a) For $s\in G$

$$s(M\oplus N) = sM\oplus sN,\ s(M\otimes_F N) = sM\otimes_F sN.$$

(*b*) For $s\in G$ and $a_1\wedge\cdots\wedge a_r\in \Lambda^r M$

$$s(a_1\wedge\cdots\wedge a_r) = s\,a_1\wedge\cdots\wedge sa_r.$$

(*c*) For each element $s\in G$ we define

$$s_M : M \longrightarrow M \text{ by } s_M(m) = sm.$$

Then, $\mathrm{Hom}_F(M,N)$ becomes a G-module if we define

$$sf = s_N f s_M^{-1}$$

for $f\in \mathrm{Hom}_F(M,N)$ and $s\in G$. In particular, for $s,t\in G$ and $f\in \mathrm{Hom}_F$-(M,N), we have

$$\begin{aligned} s(tf) &= s(t_N f t_M^{-1}) = s_N(t_N f t_M^{-1})s_M^{-1} \\ &= (s_N t_N) f (s_M t_M)^{-1} = (st)f. \end{aligned}$$

Proposition 3.3 For $f\in \mathrm{Hom}_G(M,N)$ we have $f\in \mathrm{Hom}_F(M,N)$ if and only if $sf=f$ for each $s\in G$.

Proof. That $f\in \mathrm{Hom}_G(M,N)$ means $sf=fs$ for each $s\in G$. Therefore, $sf = s_N f s_M^{-1} = s_N s_N^{-1} f = f$.

Conversely, we assume that $sf=f$ for $s\in G$. It follows that $f=sf=s_N f s_M^{-1}$ and thus we have $s_N f = f s_M$, that is, $sf=fs$. Hence f belongs to $\mathrm{Hom}_G(M,N)$. *q.e.d.*

Note that if $M_1, M_2, \cdots M_n$ are one-dimensional G-modules

$$\Lambda^r(M_1\oplus\cdots\oplus M_n) \cong \sum_{i_1<\cdots<i_r} M_{i_1}\otimes_F\cdots\otimes_F M_{i_r}$$

as G-modules, where $1\le i_1<\cdots<i_r\le n$.

We put

$$\bar{F}=\{\bar{a}\mid a\in F\}$$

and let $M^{+}=\mathrm{Hom}_F(M,\bar{F})$ denote the module of conjugate linear functionals, that is, $f\in M^{+}$ if and only if $f(a+ib)=(a-ib)f$ in case $F=\mathbf{C}$. Then M^{+} is a G-module by the relations :

(i) $sf=fs_M^{-1}$ for all $s\in G$ and $f\in M^{+}$.

(ii) $sa=a$ for all $a\in F$ and $s\in G$.

Definition 3.4 A G-module M is said to be *simple* if M has no any G-submodule, that is, no subspace N with $sN=N$ for all $s\in G$. A G-module M is *semisimple* provided it satisfies the following equivalent properties:

(i) M is a sum of simple G-submodules.

(ii) M is a direct sum of simple G-submodules.

(iii) For each G-submodule N there exists a G-submodule N' such that $M=N\oplus N'$.

Definition 3.5 Let M be an F-vector space. A *Hermitian form* on M is a function $\beta: M\times M\longrightarrow F$ such that

(i) β is linear on the first argument and conjugate linear on the second argument.

(ii) $\beta(x,y)=\overline{\beta(y,x)}$ for $x,y\in M$.

(iii) $\beta(x,x)\geqq 0$ for $x\in M$ and $\beta(x,x)=0$ if and only if $x=0$.

A Hermitian form β is said to be *G-invariant* if

$$\beta(sx,sy)=\beta(x,y)$$

for each $s \in G$ and $x, y \in M$.

Let β be a Hermitian form of M. A function $c_\beta : M \longrightarrow M^+$ is called the *correlation* associated with β if

$$c_\beta(x)(y) = \beta(x, y)$$

for $x, y \in M$.

Proposition 3.6 Let G be a compact topological group. Then, a Hermitian form of M is G-invariant if and only if c_β is a G-morphism.

Proof. We assume that β is a G-invariant Hermitian form of M. Then, for all $s \in G$ and $x, y \in M$, we have

$$c_\beta(sx)(y) = \beta(sx, y) = \beta(x, s^{-1}y) = c_\beta(x)(\bar{s}^1 y) = s(c_\beta(x)(y)),$$

because $c_\beta(x) \in M^+$ and $s(c_\beta(x)) = c_\beta(x)\bar{s}^1$ for $s \in G$. Therefore, $c_\beta(sx) = sc_\beta(x)$, and thus c_β is a G-morphism.

Conversely, we assume that C_β is a G-morphism. Then, for $x, y \in M$ and $s \in G$ we have

$$\beta(sx,\ sy) = c_\beta(sx)(sy) = s\, c_\beta(x)(sy)$$

$$= (s^{-1}(s\, c_\beta(x)))(y) = (s^{-1}s)(c_\beta(x)(y)) = \beta(x, y)$$

and thus β is G-invariant. *q. e. d.*

Note that every G-module has a G-module has a G-invariant Hermitian form if G is compact ([2],[5]).

Lemma 3.7 Every G-module M is a direct sum of simple G-submodules if G is compact (i.e., every G-module is semisimple if G is compact).

Proof. Let β be a G-invariant Hermitian form of M. If M is not simple, then there is a G-submodule L of M. We put

$$L' = \{y \in M \mid \beta(x,y) = 0 \text{ for all } x \in L\}.$$

Then L' is a G-submodule of M such that $L \cap L' = \{0\}$. We want to show that each element $a \in M$ has only one representation $a = l + l'$ such that $l \in L$ and $l' \in L'$. We define

$$L \longrightarrow F \ (y \longmapsto \beta(a,y)),$$

then $\beta(a, \quad) \in L^+$. Since $c_\beta : L \longrightarrow L^+$ is an isomorphism, there exists an element l in L such that

$$c_\beta(l) = \beta(l, \quad) = \beta(a, \quad).$$

Then, $a - l = l' \in L'$, because,

$$\beta(l', x) = \beta(a-l, x) = \beta(a,x) - \beta(l,x) = 0$$

for all $x \in L$. Hence we have $a = l + l'$ such that $l \in L$ and $l' \in L'$. Suppose that there is another representation $a = l_1 + l_1'$ such that $l_1 \in$ L and $l_1' \in L'$. Then

$$(l - l_1) + (l' - l_1') = 0 \ \text{ implies } \ l = l_1 \ \text{ and } \ l' = l_1'.$$

This means that

$$M = L \oplus L',$$

and thus M is semisimple. *q. e. d.*

Furthermore, it is known that for a G-morphism $f : M \longrightarrow N$ between F-vector spaces([5])

(i) if M is simple, then f is either zero or a monomorphism;

(ii) if N is simple, then f is either zero or an epimorphism;

(iii) if M and N are simple, then f is either zero or an isomorphism;

(iv) if $M=N$ is simple and F is algebraically closed (i.e., $F=\mathbf{C}$), then f is a multiplication by a scalar.

For (iv), note that if $\lambda \in F$ is an eigenvalue of f and $L=\mathrm{Ker}(f-\lambda)(\neq 0)$, then L is a G-submodule of M. Hence $L=M$ and $f(x)=\lambda x$ for $x \in M$.

Lemma 3.8 Let us suppose that $F=\mathbf{C}$ and G is abelian. If a G-module M over $\mathbf{C}$ is simple, then M is a one-dimensional G-module and for each $s \in G$

$$s_M : M \longrightarrow M$$

is multiplication by a complex number λ_S. Therefore, there is a group homomorphism

$$\varphi : G \longrightarrow \mathbf{C}^{\times} \quad (\varphi(s) = \lambda_S)$$

where $\mathbf{C}^{\times} = \mathbf{C} - \{0\}$.

Proof. Since G is abelian, for $s,\ t \in G$ and $m \in M$, we have

$$s_M(tm) = s(tm) = (st)(m) = t(sm) = t(s_M m),$$

that is, s_M is a G-morphism. By (iv) there exists only one complex number λ_S such that for all $m \in M$

$$s_M(m) = \lambda_S m.$$

Next, for $s,\ t \in G$ and $m \in M$, we have

$$\varphi(st)m = \lambda_{st}\, m = (st)_M\, m = s_M(t_M m) = \lambda_S(\lambda_t m) = (\lambda_S \cdot \lambda_t)\, m$$
$$= \varphi(s)\, \varphi(t)\, m$$

and thus we have $\varphi(st) = \varphi(s)\ \varphi(t)$, that is, φ is a group homomorphism. Since each one-dimensional subspace of M is a G-submodule and M is simple, M is a one-dimensional G-module. *q. e. d.*

§ 4. Representation Rings

For a topological group G, let $\mathbf{M}_F(G)$ denote the set of isomorphism classes $[M]$ of G-modules M over F. Then $\mathbf{M}_F(G)$ is a semiring with operations

$$[M]+[N]=[M\oplus N],\ [M]\cdot[N]=[M\otimes_F N],$$

where $[M]$ and $[N]$ are in $\mathbf{M}_F(G)$.

Definition 4.1 The *representation ring* $R_F(G)$ of G is the completion of the semiring $\mathbf{M}_F(G)$. That is,

$$R_F(G)=\mathbf{M}_F(G)\times\mathbf{M}_F(G)/\sim,$$

where the binary relation "$\sim$" is definded as follows:

$([M_1],[N_1])\sim([M_2],[N_2])$ iff there exists $[L]\in\mathbf{M}_F(G)$ such that $[M_1\oplus N_2\oplus L]=[N_1\oplus M_2\oplus L]$.

The equivalence class of $([M],[N])$ will be denoted by $<[M],[N]>$. In this case we put $<[M],[N]>=[M]-[N]$, where $[M]$ and $[N]$ are in $\mathbf{M}_F(G)$.

In the ring $R_F(G)$ we know the following

(a) addition :

$$<[M_1],[N_1]>+<[M_2],[N_2]> = <[M_1\oplus M_2],[N_1\oplus N_2]>$$

(b) multiplication :

$$<[M_1], [N_1]> \cdot <[M_2], [N_2]> = <[M_1 \otimes_F M_2 \oplus N_1 \otimes_F N_2],$$

$$[M_1 \otimes_F N_2 \oplus N_1 \otimes_F M_2]>$$

(c) zero element :

$$<0, 0> = <[M], [M]>$$

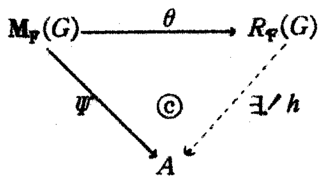
(d) identity element :

$$<[F], 0> (sa = a \ \text{ for } \ s \in G \text{ and } a \in F)$$

(e) there is the natural inclusion map

$$\theta : \mathbf{M}_F(G) \longrightarrow R_F(G) \ \ (\theta([M]) = <[M], 0> = [M])$$

such that if there is a semiring homomorphism $\Psi : \mathbf{M}_F(G) \to A$ for any ring A, then there exists a unique ring homomorphism $h : R_F(G) \longrightarrow A$ satisfying the commutative diagram:

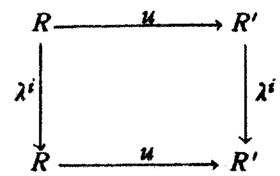


Definition 4.2 A λ-*semiring* R is a commutative semiring together with functions $\lambda^i : R \to R$ for $i \geq 0$ satisfying the conditions

(i) $\lambda^0(x) = 1$ and $\lambda^1(x) = x$ for each $x \in R$.

(ii) $\lambda^k(x+y) = \sum_{i+j=k} \lambda^i(x)\lambda^j(y)$ for $k \geq 0$ and $x, y \in R$.

A morphism $u : (R, \lambda^i) \longrightarrow (R', \lambda^i)$ is called a λ-semiring homomorphism if $u : R \longrightarrow R'$ is a semiring morphism and if the diagram



is commutative. In particular, if in a λ-semiring (R, λ^i) R is a ring, then (R, λ^i) is called a *λ-ring*.

Proposition 4.3 $R_F(G)$ has a λ-ring structure.

Proof. At fist, we define

$$\lambda^i : \mathbf{M}_F(G) \longrightarrow \mathbf{M}_F(G)$$

by $\lambda^i([M]) = [\Lambda^i M]$, where $\Lambda^i M$ is the i-th exterior algebra of M. Note that $\Lambda^i M$ is a G-module (See Section 3.).

$$\lambda^0([M]) = [\Lambda^0 M] = [F] = 1 \text{ and } \lambda^1([M]) = [\Lambda^1 M] = [M].$$

Since

$$\lambda^k([M]+[N]) = \lambda^k([M \oplus N]) = [\Lambda^k(M \oplus N)]$$
$$= [\sum_{i+j=k} \Lambda^i M \wedge \Lambda^j N] = \sum_{i+j=k} \lambda^i([M]) \cdot \lambda^j([N]),$$

$(\mathbf{M}_F(G), \lambda^i)$ is a λ-semiring.

We define

$$\lambda_t : \mathbf{M}_F(G) \longrightarrow 1 + R_F(G)\,[[\,t\,]]^+$$

by $\lambda_t([M]) = 1 + \sum_{i=1}^{\infty} \lambda^i([M]) t^i$ where for a ring R

$$1 + R[[\,t\,]]^+ = \{1 + a_1 t^1 + a_2 t^2 + \cdots \mid a_i \in R,\ i = 1, 2, \cdots\}.$$

Note that $\lambda_t([M]) = \sum_{i=0}^{\infty} \lambda^i([M]\, t^i \in 1 + \mathbf{M}_F(G)\,[[\,t\,]]^+ \subset 1 + R_F(G)\,[[\,t\,]]^+$, and that the subset $1 + R_F(G)\,[[\,t\,]]^+$ of the ring $R_F(G)\,[[\,t\,]]$ is a multiplicative group. By (e) we have the group homomorphism

$\tilde{\lambda}_t : R_F(G) \longrightarrow 1 + R_F(G)\,[[\,t\,]]^+$ satisfying the commutative diagram:

$$\begin{array}{ccc} \mathbf{M}_F(G) & \xrightarrow{\ \theta\ } & R_F(G) \\ & \lambda_t \searrow \quad © \quad \swarrow \tilde{\lambda}_t & \\ & 1 + R_F(G)\,[[\,t\,]]^+ , & \end{array}$$

because

$$\lambda_t([M]+[N]) = \lambda_t([M\oplus N]) = 1+\sum_{i=1}^{\infty} \lambda^i([M\oplus N])t^i$$

$$=1+\sum_{k=1}^{\infty} \sum_{i+j=k} \lambda^i([M])\, \lambda^j([N])\, t^k = \lambda_t([M])\, \lambda_t([N]),$$

that is, λ_t is a group homomorphism. We put for $x \in R_F(G)$

$$\tilde{\lambda}_t(x) = \sum_{i=0}^{\infty} \tilde{\lambda}^i(x)\, t^i,$$

then $\tilde{\lambda}^0(x) = 1$, $\tilde{\lambda}^1(x) = x$ and $\tilde{\lambda}^k(x+y) = \sum_{i+j=k} \tilde{\lambda}^i(x)\, \tilde{\lambda}^j(y)$ where $x,y \in R_F(G)$. Therefore, $(R_F(G), \tilde{\lambda}^i)$ is a λ-ring. *q.e.d.*

By Lemma 3.7 it follows that the representation $R_F(G)$ is generated by all simple G-module classes.

Definition 4.4. For each $[M] \in \mathbf{M}_F(G)$ we define

$$\begin{array}{ccc} \chi_M : G & \longrightarrow & F \\ \Downarrow & & \Downarrow \\ s & \longmapsto & \mathrm{Tr}\, s_M \end{array}$$

which is called the *character* of M, where $\mathrm{Tr}\, s_M$ is the trace of the matrix s_M. We put

$$\widetilde{\mathbf{Ch}}_F(G) = \{\, \chi_M \mid [M] \in \mathbf{M}_F(G) \,\}.$$

By the elementary properties of Tr it follows that

(i) $\chi_{M\oplus N} = \chi_M + \chi_N$

(ii) $\chi_{M\otimes N} = \chi_M\, \chi_N$.

The set $\mathbf{C}_F(G)$ of continuous maps $f: G \to F$ is a normed space by $\|f\| = \sup_{s\in G} \| f(s) \|$ for each $f \in \mathbf{C}_F(G)$. Moreover, $\mathbf{C}_F(G)$ is a commutative ring with 1. In this case, since $\widetilde{\mathbf{Ch}}_F(G) \subset \mathbf{C}_F(G)$, let

$\mathbf{Ch}_F(G)$ be the subring of $\mathbf{C}_F(G)$ generated by $\widetilde{\mathbf{Ch}}_F(G)$.

Note that there exists the isomorphism

$$\varphi : R_F(G) \longrightarrow \mathbf{Ch}_F(G)$$

defind by

$$\varphi([M]-[N]) = \chi_M - \chi_M$$

where $[M]-[N] \in R_F(G)$.

Proposition 4.5 (i) Let M and N be G-modules over F. Then $\chi_M = \chi_N$ implies that $M \cong N$ as G-modules.

(ii) If $u : G \longrightarrow H$ is a topological group homomorphism such that for each $t \in H$ there exists $s \in H$ with $sts^{-1} \in u(G)$, then

$$\mathbf{R}_F(u) : R_F(H) \longrightarrow R_F(G)$$

is a monomorphism, where $\mathbf{R}_F(u)([M]) = [u^*M]$ for $[M] \in R_F(H)$, $u^*M = M$ as sets, and $g(u^*M) = u(g)M$ for each $g \in G$.

(iii) If $u : G \longrightarrow G$ is an inner automorphism, then $\mathbf{R}_F(u)$ is the identity of $R_F(G)$.

Proof. (i) If $\chi_M = \chi_N$ implies that $M \not\cong N$, then the above isomorphism φ does not make sense. Hence $\chi_M = \chi_N$ implies $M \cong N$.

(ii) For each $[M] \in R_F(H)$ we have $\chi_{u^*M} = \chi_M \cdot u$ on G. Hence $\chi_M \circ u = \chi_N \cdot u$ implies $\chi_M = \chi_N$ where $[N] \in R_F(H)$, because by our hypothesis for each $g \in G$ there exists s and t in H such that $u(g) = sts^{-1}$ and by $\chi(sts^{-1}) = \chi(t)$.

Therefore, by (i) $M\cong N$ and thus $[M]=[N]$. Hence, we have

$$\mathbf{R}_F(u)([M])=\mathbf{R}_F(u)([N])$$

which implies that $\mathbf{R}_F(u)$ is a monomorphism.

(iii) By our assumption there is an element $s\in G$ such that $u(t)=sts^{-1}$ for $t\in G$. For each element $[M]\in R_F(G)$, we have

$$t_{u^*M}(u^*M)=s_M t_M s_M^{-1} M$$

and thus

$$\chi_{u^*M}(t)=\mathrm{Tr}\ t_{u^*M}=\mathrm{Tr}\,(s_M t_M s_M^{-1})=\mathrm{Tr}\ t_M=\chi_M(t).$$

That is, $\chi_{u^*M}=\chi_M$. By (i) we have $u^*M\cong M$, and hence,

$$\mathbf{R}_F(u)([M])=[u^*M]=[M]. \quad q.e.d.$$

§5. Bundles with G-structures

Definition 5.1. Let $\xi=(E(\xi),\pi_\xi,X)$ be an F-vector bundle over X. If there are a locally trivial principal G-bundle $\eta=(P,\pi_\eta,X)$ over X and a G-module $\hat{M}$ over F such that

$$E(\xi)\cong P\times_G M$$

where $(p,m)s=(ps,s^{-1}m)$ for $p\in P$, $m\in M$ and $s\in G$ and $P\times_G M=(P\times M)/G$, that is, $\xi=\eta[M]$, then ξ is called *a vector bundle with G-structure by* η.

Definition 5.2 Let X be a paracompact space. Let $\mathrm{Vect}_F{}^G\eta(X)$ be the set of all isomorphism classes of F-vector bundles over with G-structure by η, where η is a fixed principal G-bundle over X which is locally trivial.

Then $\mathrm{Vect}_F{}^G\eta(X)$ is a semiring with operations

$$[\mu]+[\nu]=[\mu\oplus\nu],\ [\mu][\nu]=[\mu\otimes_F\nu]$$

where $[\mu]$ and $[\nu]$ are in $\mathrm{Vect}_F{}^G\eta(X)$. In fact, if $\mu=\eta[M]$ and $\nu=\eta[N]$, then

$$\mu\oplus\nu=\eta[M\oplus N],\ \ \mu\otimes_F\nu=\eta[M\otimes_F N]$$

Moreover, if $K_F{}^G\eta(X)$ denotes the completion of $\mathrm{Vect}_F{}^G\eta(X)$, then the ring $K_F{}^G\eta(X)$ is called the *K-ring* of *F*-vector bundls over *X* with *G*-structure by η.

Let $\mathbf{T}_{Gr}$ be the category consisting of all topological groups and topological group homomorphisms and λ-Ring the category of λ-rings and λ-ring homomorphisms.

Proposition 5.3. (i) $\mathbf{R}_F:\mathbf{T}_{Gr}\longrightarrow\lambda$-Ring is a cofunctor.

(ii) $K_F{}^G\eta(X)$ is a λ-ring, where η is a locally trivial principal G-bundle over X.

Proof. (i) For each $u:G\longrightarrow H$ in $\mathbf{T}_{Gr}$

$$\mathbf{R}_F(u):R_F(H)\longrightarrow R_F(G),\ (\mathbf{R}_F(u)([M])=[u^*M],\ [M]\in R_F(H))$$

is defined as in (ii) of Proposition 4.5. By Proposition 4.3, $R_F(G)$ is a λ-ring for each $G\in\mathbf{T}_{Gr}$. It is easy to prove that

(a) for $1_G:G\to G$ in $\mathbf{T}_{Gr}$ $\quad \mathbf{R}_F(1_G)=1_{\mathbf{R}_{F(G)}}$

(b) for $u:G\to H$ and $v:H\to L$ in $\mathbf{T}_{Gr}$

$$\mathbf{R}_F(v\circ u)=\mathbf{R}_F(u)\,\mathbf{R}_F(v)$$

(c) for $[M]$ and $[N]\in R_F(H)$

$$\mathbf{R}_F(u)([M]+[N]) = \mathbf{R}_F(u)([M]) + \mathbf{R}_F(u)([N])$$

$$\mathbf{R}_F(u)([M][N]) = \mathbf{R}_F(u)([M \otimes_F N]) = [\mathbf{R}_F(u)([M]) \otimes_F \mathbf{R}_F(u)([N])$$

$$= \mathbf{R}_F(u)([M])\mathbf{R}_F(u)([N]).$$

Since for each $[M] \in R_F(H)$

$$\mathbf{R}_F(u)(\lambda^i[M]) = \mathbf{R}_F(u)([\Lambda^i M]) = [\Lambda^i u^* M] = \lambda^i([u^* M])$$

$$= \lambda^i \mathbf{R}_F(u)([M]),$$

$\mathbf{R}_F(u)$ is a λ-ring homomorphism.

Therefore, $\mathbf{R}_F$ is a cofunctor.

(ii) As in Proposition 4.3 we can make $K_F{}^G\eta(X)$ a λ-ring.

That is, we define

$$\begin{array}{ccc} \lambda^i : \mathrm{Vect}_F{}^G\eta(X) & \longrightarrow & \mathrm{Vect}_F{}^G\eta(X) \\ \Downarrow & & \Downarrow \\ [\xi] & \longmapsto & [\Lambda^i \xi] \end{array}$$

then $\mathrm{Vect}_F{}^G\eta(X)$ is a λ-semiring. (Note that if $\xi = \eta[M]$, then $\Lambda^i \xi = \eta[\Lambda^i M]$.) Since $K_F{}^G\eta(X)$ is the completion of $\mathrm{Vect}_F{}^G\eta(X)$, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Vect}_F{}^G\eta(X) & \xrightarrow{\ \theta\ } & K_F{}^G\eta(X) \\ & \lambda_t \searrow \quad © \quad \swarrow \tilde{\lambda}_t & \\ & 1+K_F{}^G\eta(X)[[t]]^+ & \end{array}$$

where θ is the natural inclusion map and λ_t is defined by

$$\begin{array}{ccc} \lambda_t : \mathrm{Vect}_F{}^G\eta(X) & \longrightarrow & 1+K_F{}^G\eta(X)[[t]]^+ \\ \Downarrow & & \Downarrow \\ [\xi] & \longmapsto & 1+\sum_{i \geq 1} \lambda^i([\xi])t^i. \end{array}$$

Since λ_t is a group homomorphism, so is $\widetilde{\lambda}_t$. Let us put $\widetilde{\lambda}_t(x)=1+\sum_{i=1}^{\infty}\widetilde{\lambda}^i(x)t^i$ for each $x\in K_F{}^G\eta(X)$. Then, it follows that $(K_F{}^G\eta(X), \widetilde{\lambda}_i)$ is a λ-ring .q.e.d.

Let $K^G(X)$ denotes the set of all isomorphism classes of locally trivial principal G-bundles over X. Also, by $\dot{\cup}$ we mean disjoint union.

Theorem 5.4. (i) We assume that a topological group G and a locally trivial principal G-bundle η are fixed. Then for any compact space X, there is the λ-ring isomorphism

$$\begin{array}{ccc} \varphi: R_F(G) & \longrightarrow & K_F{}^G\eta(X) \\ \cup\!\!\!| & & \cup\!\!\!| \\ [M]-[N] & \leadsto & [\eta[M]]-[\eta[N]]. \end{array}$$

(ii) If we put

$$\dot{\bigcup_{f\in[X,B_G]}} R_F(G)_f=\mathcal{R}_F(G), \qquad \dot{\bigcup_{[\eta]\in K^G(X)}} K_F{}^G\eta(X)=\mathcal{K}_F{}^G(X),$$

then there is an injection.

$$\Psi:\mathcal{R}_F(G)\longrightarrow \mathcal{K}_F{}^G(X)$$

where $R_F(G)=R_F(G)_f$ for all $f\in[X,B_G]$ and X is paracompact.

Proof. (i) It is clear that φ is injective by our definition. Take an element $[\mu]-[\nu]\in K_F{}^G\eta(X)$, then there are two G-modules M and N such that

$$\eta[M]=\mu,\ \eta[N]=\nu.$$

Hence,

$$\varphi([M]-[N])=[\mu]-[\nu].$$

Moreover, for $[M]\in R_F(G)$, we have

$$\varphi(\lambda^i([M]))=\varphi([\Lambda^i M])=[\eta[\Lambda^i M]]$$
$$=[\Lambda^i(\eta[M])]=\lambda^i\varphi([M]).$$

This implies that φ is a λ-ring homomorphism.

(ii) Since $\omega_G=(E(\omega_G),\pi_G,B_G)$ is a universal principal G-bundle, there is a bijection

$$u:[X,B_G]\longrightarrow K^G(X)$$

([8], [9]). For $f\in[X,B_G]$, assume that $u(f)=[\eta]$. Then, for $[M]-[N]\in R_F(G)_f$, we define

$$\Psi([M]-[N])=[\eta[M]]-[\eta[N]]\in K_F{}^G\eta(X).$$

By (i) it follows that Ψ is an injection. *q.e.d.*

Example 5.5. As is well known, a compact Lie group has maximal tori ([1], [2]), and a maximal torus of $SO(2n)$ is a subgroup of all diagonal matrices $D(\theta_1,\cdots,\theta_n)$, where

$$D(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

and $D(\theta_1,\cdots,\theta_n)$ is the $2n\times 2n$-matrix with $D(\theta_1),\cdots,D(\theta_n)$ on the diagonal. Similarly, a maximal torus of $SO(2n+1)$ is the subring of all diagonal matrices $D(\theta_1,\cdots,\theta_n,*)$, where

$$D(\theta_1,\cdots,\theta_n,*)=\begin{pmatrix}D(\theta_1) & & & 0\\ & \ddots & & \\ & & D(\theta_n) & \\ 0 & & & 1\end{pmatrix}$$

(Note that a maximal torus is a compact connected abelian Lie group).

Let G be a compact Lie group, and let T^n be a maximal torus of G. By Lemma 3.7 and Lemma 3.8 every simple T^n-module over C is a one-dimensional C-vector space $M(k_1, \cdots, k_n)$ ($\dim_R T^n = n$ and $k_i \in Z$ for $i = 1, 2, \cdots, n$) and for $(\theta_1, \cdots, \theta_n) \in T^n$, $z \in M(k_1, \cdots, k_n)$

$$(\theta_1, \cdots, \theta_n) z = \exp[2\pi i (k_1 \theta_1 + \cdots + k_n \theta_n)]$$

([1]).

In particular, since $s(x \otimes y) = sx \otimes sy$ for T^n-modules x, y and $s \in G$, we have

$$M(k_1, \cdots, k_n) \otimes M(l_1, \cdots, l_n) = M(k_1 + l_1, \cdots, k_n + l_n).$$

Furthermore we can prove that

$$R_C(T^n) = Z[\alpha_1, \bar{\alpha}_1^1, \cdots, \alpha_n, \bar{\alpha}_n^1],$$

where $\alpha_1 = M(1, 0, \cdots 0), \cdots, \alpha_n = M(0, \cdots 0, 1)$.

Thus, for any locally trivial principal T^n-bnudle η, we have

$$K_C^{T^n} \eta(X) \cong Z[\alpha, \bar{\alpha}_1^1, \cdots, \alpha_n, \bar{\alpha}_n^1].$$

Since the inclusion map $T^n \longrightarrow G$ satisfies the condition of (ii) of Proposition 4.5, $R_C(G)$ is a subring of $R_C(T^n) = Z[\alpha_1, \bar{\alpha}_1^1, \cdots \alpha_n, \bar{\alpha}_n^1]$.

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