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ON THE EXISTENCE PROBLEM OF A CERTAIN PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

For a point $x=(x^1, x^2, \cdots, x^n)$ in the n-dimensional Euclidean space E^n, we denote its norm by $\|x\|$. Throughout this note we will denote by B the open unit disk

$$\{(u,v) : u^2+v^2<1\}$$

in E^2, the boundary of B will be denoted by ∂B and the closure by $\bar{B}$.

Consider two continuous functios $\xi_i: \partial B \to E^n$, $i=1,2$. Let φ be a sense preserving homeomorphism of ∂B onto itself and define

$$\delta(\varphi)=\max_{\theta}\|\xi_1(\theta)-\xi_2(\varphi(\theta))\|.$$

The Frechét distance $d(\xi_1, \xi_2)$ between ξ_1 and ξ_2 is then defined to be

$$d(\xi_1,\xi_2)=\inf_{\varphi}\delta(\varphi)$$

where the infimum is taken over all sense preserving homeomorphisms ∂B onto itself. The distance so defined is non-negative, symmetric and satisfies the triangle inequality.

If $d(\xi_1, \xi_2)=0$, then we say two functions ξ_1 and ξ_2 are Frechét equivalent. A Frechét curve Γ is an equivalence class of functions in $C(\partial B)$, i.e. continuous on ∂B, under Frechét equivalence. But we will delete the word Frechét and simply call it curve. An element of the equivalence class is called a representation of the curve.

A curve Γ is called a Jordan curve if it has a representation which is a homeomorphism and such a representation is called topological. Further, it is customary to use the same term, Jordan curve, to mean a subset of E^n which is the common graph of representations of Γ.

If Γ_1 and Γ_2 are two curves in E^n and if ξ_1 and ξ_2 are any representations of Γ_1 and Γ_2, respectively, then the Frechét distance between Γ_1 and Γ_2 is defined by $d(\Gamma_1, \Gamma_2)=d(\xi_1, \xi_2)$.

Suppose Γ and Γ_n, $n=1,2,\cdots$, are curves. If $d(\Gamma, \Gamma_n)\to 0$ as $n\to\infty$, then we say Γ_n converges to Γ in the sense of Frechét.

A surface in E^n is a continuous map $\zeta=(\zeta^1,\cdots,\zeta^n)$ of $\bar{B}$ into E^n and $\zeta|\partial B$ is called the boundary of the surface ζ. If $\zeta|\partial B$ is a representation of a Jordan curve Γ, then we say ζ is bounded by Γ. All the notions above for Frechét curves can be analogously defined but we omit the details and simply refer to [8].

For a surface $\zeta(u,v)=(\zeta^1(u,v), \cdots, \zeta^n(u,v))$, $(u,v)\in\bar{B}$, we will consistently use the following notations:

$$E=\zeta_u\cdot\zeta_u=\zeta_u^2=\sum_{i=1}^{n}(\zeta_u^i)^2$$

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$$F=\zeta_u\cdot\zeta_v$$
$$G=\zeta_v\cdot\zeta_v=\zeta_v^2$$

The problem we wish to investigate in this note is the following:

Given a Jordan curve Γ in E^n, we wish to show the existence of a surface ζ satisfying the following conditions.

(1) ζ^j, $j=1,2,\cdots,n$, are continuous on $\bar{B}$ and harmonic in B

(2) ζ satisfies the partial differential equations $E=G$ and $F=0$

(3) $\zeta|\partial B$ is a topological representation of Γ.

This problem was first solved in 1930, independently by J. Douglas [2] and T. Rado [4]. Their work on the problem was later simplified by R. Courant [1]. We shall give an independent solution to this problem by cosidering it as a one dimentional variational problem using a method of trigonometrical series where by giving some basis for obtaining an approximation.

The following results, most of which can be found in [6], will be needed for the subsequent argument.

Let us denote by $C_C^\infty(B)$ the class of real functions g which are of class C^∞ in B with a compact support in B. Elements of $C_C^\infty(B)$ will be called test functions. A function $f:B\to E^1$ is said to belong to the class $H_2^1(B)$ if $f\in L_2(B)$ and if there exist real functions $p_\alpha\in L_2(B)$, $\alpha=1,2$, such that

$$\iint_B g(u,v)p_\alpha(u,v)\ du\,dv=-\iint_B g_\alpha(u,v)f(u,v)\ du\,dv$$

for all test functions g where $g_1=g_u$ and $g_2=g_v$. The functions p_α are uniquely determined up to null functions and if $f\in H_2^1(B)$ and $f^*(u,v)=f(u,v)$ a.e in B, then $f^*\in H_2^1(B)$ and the same functions p_α will work for f^*. We call p_α (determined up to null functions) the distribution derivative of f. If f has first partials which are L_2-integrable in B, then the distribution derivatives will be the corrseponding partial derivatives.

If $\zeta=(\zeta^1,\cdots,\zeta^n)$ is a vector function of B into E^n, then we say ζ is of class $H_2^1(B)$ in case each of its components is of class $H_2^1(B)$.

For a surface $\zeta\in C(\bar{B})\cap H_2^1(B)$, the Dirichlet integral of ζ is defined by

$$D(\zeta)=\iint_B(\zeta_u^2+\zeta_v^2)\ du\,dv=\iint_B(E+G)\ du\,dv.$$

In section 2, we shall show that among all the surfaces $\zeta\in C(\bar{B})\cap H_2^1(B)$ there is one which minimizes the Dirichlet integral that is, moreover, a harmonic surface and then, in section 3, we shall prove that this minimizing surface satisfies the conditions (2) and (3) of the problem.

The following theorem due to Morrey [7] is essential.

THEOREM. *If S is an open nondegenerate Frechét surface of finite Lebesgue area, then S has at least one representation of class $C(\bar{B})\cap H_2^1(B)$ which satisfies the condition $E=G$, $F=0$.*

2. Existence Theorem

At the outset, we assume that there exists at least one surface bounded by Γ with a finite Dirichlet integral. Note, by the Dirichlet principle, that if $\zeta \in C(\bar{B}) \cap H_2^1(B)$ and if h is a surface harmonic in B such that $h|\partial B$ coincides with $\zeta|\partial B$, then $D(h) \leq D(\zeta)$. Hence in order to find a surface of minimum Dirichlet integral, we may restrict ourselves to the class of harmonic surfaces which are bounded by Γ. Our existence proof depends on the following well known lemmas.

LEMMA 1. [6] *Suppose ζ and ζ_n, $n=1,2,\cdots$, are surfaces in the class $C(\bar{B}) \cap H_2^1(B)$. If ζ_n converges to ζ on B and if the convergence is uniform in every closed subdomain of B, then*

$$D(\zeta) \leq \liminf_n D(\zeta_n).$$

LEMMA 2. [1] *The Dirichlet integral is invariant under a conformal map.*

Let $\{\zeta_n\}$ be a sequence of surfaces. We say $\{\zeta_n|\partial B\}$ satisfies the three point condition if, for some convenient points $p_{i_n} \in \partial B$ $(i=1,2,3)$ and for some positive constant m, we have

$$\|p_{i_n} - p_{j_n}\| > m, \quad \|\zeta_n(p_{i_n}) - \zeta_n(p_{j_n})\| > m, \quad i=j, \quad i,j=1,2,3$$

for all n.

LEMMA 3. [8] *Let S_n, $n=1,2,\cdots$, be a sequence of Frechét surfaces such that ∂S_n converges to a Jordan curve Γ in the sense of Frechét. Suppose $\zeta_n \in C(\bar{B}) \cap H_2^1(B)$ is a representation of S_n with $D(\zeta_n) < k$ (k a constant) and suppose that $\{\zeta_n|\partial B\}$ satisfies the three point condition. Then $\{\zeta_n|\partial B\}$ is equicontinuous.*

From this lemma we note that if $\{h_n\}$ is a sequence of harmonic surfaces bounded by Γ with $D(h_n) < k$ for some constant k and if $\{h_n|\partial B\}$ satisfies the three point condition, then, by the maximum principle, $\{h_n\}$ is sequentially compact with respect to the uniform convergence.

Now suppose Γ is given in some initial topological representation $g(t) = (g^1(t), g^2(t), \cdots, g^n(t))$ from which its general topological representation may be obtained by a relation $t = \xi(\theta)$ defining a one-to-one and continuous map from ∂B onto itself. Let us denote by Ω the class of all those maps ξ of ∂B onto itself which are one-to-one, continuous and leave three distinct points p_1, p_2, p_3 of ∂B fixed. Thus if $\xi \in \Omega$, then the composite map $g \circ \xi$ is a topological representation of Γ which maps p_1, p_2, p_3 of ∂B into three distinct fixed points $g(p_1)$, $g(p_2)$, $g(p_3)$ of Γ.

A disadvantage in dealing with the class Ω is that it is not closed under uniform convergence. Thus we consider the class Δ of those maps ξ of ∂B onto itself which are continuous, monotone and leave three distinct points p_1, p_2, p_3 of ∂B fixed. Thus Ω is a subclass of Δ and it is clear that Δ is closed under uniform convergence. And we note that, for each $\xi \in \Delta$, we get a representation $g \circ \xi$ and the class of all such representations has the three point property.

Consider now, for each $\xi \in \Delta$, the corresponding representation

$$g \circ \xi = (g^1 \circ \xi, \; g^2 \circ \xi, \; \cdots, \; g^n \circ \xi) \tag{2.1}$$

and expand in a Fourier series

$$(g^j \circ \xi)(\theta) \sim \frac{a_{j_0}(\xi)}{2} + \sum_{k=1}^{\infty} (a_{j_k}(\xi) \cos k\theta + b_{j_k}(\xi) \sin k\theta). \tag{2.2}$$

$j=1,2,\cdots,n$, where

$$a_{jk}(\xi)=\frac{1}{\pi}\int_0^{2\pi}(g^j\circ\xi)(\theta)\cos k\theta\, d\theta$$

$$b_{jk}(\xi)=\frac{1}{\pi}\int_0^{2\pi}(g^j\circ\xi)(\theta)\sin k\theta\, d\theta$$

If h_ξ^j, $j=1,2,\cdots,n$, are the harmonic functions determined by the boundary values $g^j\circ\xi$, $j=1,2,\cdots,n$, respectively, then they are given by

$$h_\xi^j(u,v)=\frac{a_{j0}(\xi)}{2}+\sum_{k=1}^{\infty}r^k(a_{jk}(\xi)\cos k\theta+b_{jk}(\xi)\sin k\theta),$$

$0\leq r<1$, where $u=r\cos\theta$ and $v=r\sin\theta$.

For notational convenience, we shall write

$$a_k=(a_{1k},\ a_{2k},\ \cdots,\ a_{nk})$$
$$b_k=(b_{1k},\ b_{2k},\ \cdots,\ b_{nk})$$
$$a_k^2=\sum_{j=1}^{n}a_{jk}^2,\quad b_k^2=\sum_{j=1}^{n}b_{jk}^2,\quad a_kb_k=\sum_{j=1}^{n}a_{jk}b_{jk}$$
$$(g\circ\xi)(\theta)\sim\frac{a_0(\xi)}{2}+\sum_{k=1}^{\infty}(a_k(\xi)\cos k\theta+b_k(\xi)\sin k\theta)$$

and the corresponding expression for the harmonic surface

$$h=(h^1,\ h^2,\ \cdots,\ h^n) \tag{2.3}$$

Let H denote the set of harmonic surfaces (2.3) determined by all $\xi\in\Delta$. We note that if $\xi\in\Omega$, then the corresponding harmonic surface h_ξ is bounded by Γ and that $h_\xi|\partial B=g\circ\xi$ so that it is a topological representation of Γ. We also note that if h is an arbitrary harmonic surface (not necessarily in H) bounded by Γ, then we can find a conformal map $T:B\to B$ such that the restriction of $\zeta=h\circ T$ on ∂B satisfies the three point condition. Thus if h_0 denotes the harmonic surface determined by $\zeta|\partial B$, then $h_0\in H$ and, by Lemma 2 and by the Dirichlet principle,

$$D(h_0)\leq D(\zeta)=D(h).$$

Thus in order to minimize the Dirichlet integral over the class of harmonic surfaces bounded by Γ, it is sufficient to consider the problem of minimizing it over the class H.

Now, for each $h\in H$, consider the Dirichlet integral

$$D(h)=\iint_B(E+G)\,du\,dv \tag{2.4}$$

In polar coordinate this becomes

$$D(h)=\int_0^{2\pi}\int_0^1\left(h_r^2+\frac{1}{r^2}h_\theta^2\right)r\,dr\,d\theta \tag{2.5}$$
$$=\pi\sum_{k=1}^{\infty}k(a_k^2(\xi)+b_k^2(\xi))$$

Since $D(h)$ is determined by $g\circ\xi$ and since g is held fixed, we let

$$P(\xi)=J(g\circ\xi)=\pi\sum_{k=1}^{\infty}k(a_k^2(\xi)+b_k^2(\xi)) \tag{2.6}$$

and consider the equivalent problem of minimizing P over the class Δ.

Note that, by Lemma 1, the funcional P of (2.6) is also lower semicontinuous on the class Δ with respect to uniform convergence: in fact, if $\{\xi_m\}$ is a sequence of elements of Δ which converges uniformly to an element $\xi\in\Delta$ and if h_{ξ_m} and h_ξ are the corresponding harmonic surfaces, then by the maximum principle, h_{ξ_m} converges uniformly to h_ξ so that

$$P(\xi)=D(h_\xi)\leq \liminf_m D(h_{\xi_m})=\liminf_m P(\xi_m).$$

Also note that if a sequence $\{g\circ\xi_m\}$, $\xi_m\in\Delta$, converges uniformly to a representation τ of Γ, then ξ_m converges uniformly to $g^{-1}(\tau)$ and this belongs to the class Δ.

Now since we are dealing with the problem of minimizing P, we may assume that there is a constant k such that $P(\xi)\leq k$ for all $\xi\in\Delta$. By Lemma 3 and by the above remark, Δ is compact with respect to the topology of uniform convergence. Since P is lower semicontinuous, it follows that P attains a minimum on Δ.

3. Property of the Minimizing Surface

Let ξ^* denote the element of Δ which minimizes P and let h^* be the corresponding harmonic surface. In this section we shall show that h^* satisfies the condition $E=G$ and $F=0$.

Because there is no a priori reason to believe that $h^*|\partial B$ is differentiable we will require the following theorem, known as the sewing theorem.

SEWING THEOREM. [1] *Suppose that the plane domain* G_+ *(consisting of the exterior of an analytic curve* C_+*) and* G_- *(consisting of the interior of another analytic closed curve* C_-*) are combined into one region* G *by an analytic transformation* $z_+=t(z_-)$ *that establishes a biunique correspondence between all the points* A_+ *on* C_+ *and* A_- *on* C_-*. Then the domain* G_+ *and* G_- *can be mapped conformally onto two domain* G'_+ *and* G'_-*, respectively, in such a way that* C_+ *and* C_- *are transformed into the same curve* C'*, that* G_+ *is transformed into the exterior of* C' *and* G_- *into the interior of* C' *and, further, that corresponding poits* A_+ *and* A_- *go into the same point* A' *on* C'*. In other words, by a conformal mapping two separated components* G_+ *and* G_- *can be fitted together into a single domain* G'*:*

Let r_0, $0<r_0<1$, be an arbitrary but fixed number and cut the unit disk B along the circle ∂B_{r_0}, the circle with center at the origin and radius r_0. If λ is a real analytic function and is periodic of period 2π, then there exists an ε_0 such that the function $\varphi_\varepsilon(\theta)=\theta+\varepsilon\lambda(\theta)$ for $|\varepsilon|\leq\varepsilon_0$ is analytic and biunique. Let $B^*(\varepsilon)$ denote the domain obtained by identifying the point (r_0, θ) on the outer edge of the cut with the point $(r_0{:}\theta+\varepsilon\lambda(\theta))$ on the inner edge. Then, by the sewing theorem, there exists a conformal map T^* from $B^*(\varepsilon)$ onto B. In order to see this, let G_- be the domain obtained from B_{r_0} by the transformation $(r, \theta)\to(r, \theta+\varepsilon\lambda(\theta))$ and let G_+ be the domain consisting of the exterior of ∂B_{r_0}. If we denote by G the domain consisting of G_+ and G_-, then, by the sewing theorem, there exists a conformal map T from G onto the whole plane. Let C be the image of ∂B under T and let T' be the conformal map from the domain bounded by C onto B. Then $T^*=T'\circ T$ is the conformal map from $B^*(\varepsilon)$ onto B.

Now consider the function $h^*(r_0, \ \theta+\varepsilon\lambda(\theta))$, $|\varepsilon|\leq\varepsilon_0$, which is defined on ∂B_{r_0}, and denote by h_ε the function harmonic in B_{r_0} determined by the boundary value $h^*(r_0, \theta+\varepsilon\lambda(\theta))$. And define a function ζ_ε by

$$\zeta_\varepsilon(r, \theta)=\begin{cases} h_\varepsilon(r, \theta) & \text{if } (r, \theta)\in\bar{B}_{r_0} \\ h^*(r, \theta) & \text{if } (r, \theta)\in\bar{B}-\bar{B}_{r_0}\end{cases}$$

Let $\zeta_\varepsilon^* = T^* \circ \zeta_\varepsilon$, then ζ_ε^* is continuous and piecewise smooth on B, and by Lemma 2, we have $D(\zeta_\varepsilon^*) = D(\zeta_\varepsilon)$. Since h^* minimizes the Dirichlet integral, it follows that

$$D(h^*) \leq D(\zeta_\varepsilon^*). \tag{3.1}$$

Thus if we denote by D_0 the Dirichlet integral over the disk B_{r_0}, then (3.1) and Lemma 2 imply

$$D_0(h_\varepsilon) \geq D_0(h^*) \tag{3.2}$$

because ζ_ε and h^* coincide in $\bar{B} - \bar{B}_{r_0}$. We shall use this relation to find the characteristic property of h^*

Consider the Fourier series expansion of the function $h^*(r_0, \theta + \varepsilon\lambda(\theta))$ which is defined on ∂B_{r_0}

$$h^*(r_0, \theta + \varepsilon\lambda(\theta)) = \frac{a_0(\varepsilon)}{2} + \sum_{k=1}^{\infty} (a_k(\varepsilon) \cos k\theta + b_k(\varepsilon) \sin k\theta)$$

where $a_k(\varepsilon)$ and $b_k(\varepsilon)$ are the Fourier coefficients of the function $h^*(r_0, \theta + \varepsilon\lambda(\theta))$ which depend upon ε.

Then the function h_ε harmonic in B_{r_0} with the boundary value $h^*(r_0, \theta + \varepsilon\lambda(\theta))$ is given by

$$h_\varepsilon(r, \theta) = \frac{a_0(\varepsilon)}{2} + \sum_{k=1}^{\infty} \left(\frac{r}{r_0}\right)^k (a_k(\varepsilon) \cos k\theta + b_k(\varepsilon) \sin k\theta) \tag{3.3}$$

$$0 \leq r < r_0$$

Hence the Dirichlet integral of (3.3) over the disk B_{r_0}, which depends on ε, is given by

$$\begin{aligned} \Psi(\varepsilon) = D_0(h_\varepsilon) &= \int_0^{2\pi} \int_0^{r_0} \left(h_{\varepsilon,r}^2 + \frac{1}{r^2} h_{\varepsilon,\theta}^2\right) r\,dr\,d\theta \\ &= \pi \sum_{k=1}^{\infty} k(a_k^2(\varepsilon) + b_k^2(\varepsilon)) \end{aligned} \tag{3.4}$$

where $h_{\varepsilon, r}$ and $h_{\varepsilon, \theta}$ denote the partial derivatives of h_ε with respect to r and θ, respectively. Using an elementary argument, it is easy to show that (3.4) can be differentiated termwise.

Since D_0 attains minimum at h^*, the function Ψ attains minimum at $\varepsilon = 0$. Hence $\Psi'(0) = 0$. Thus after differentiating Ψ term-by-term with respect to ε and putting $\varepsilon = 0$, we get

$$\begin{aligned} \sum_{k=1}^{\infty} k \Big(a_k(0) &\int_0^{2\pi} h_\theta^*(r_0, \theta) \lambda(\theta) \cos k\theta \, d\theta \\ &+ b_k(0) \int_0^{2\pi} h_\theta^*(r_0, \theta) \lambda(\theta) \sin k\theta \, d\theta\Big) = 0 \end{aligned} \tag{3.5}$$

Note that

$$h^*(r_0, \theta) = \frac{a_0^*}{2} + \sum_{k=1}^{\infty} r_0^k (a_k^* \cos k\theta + b_k^* \sin k\theta)$$

where $a_k^* = a_k(\xi^*)$ and $b_k^* = b_k(\xi^*)$, and also note that this series converges uniformly and absolutely since $0 < r_0 < 1$. Thus integrating termwise, we get

$$\begin{aligned} a_k(0) &= \frac{1}{\pi} \int_0^{2\pi} h^*(r_0, \theta) \cos k\theta \, d\theta \\ &= \frac{1}{\pi} \int_0^{2\pi} \left\{ \frac{a_0^*}{2} + \sum_{n=1}^{\infty} r_0^n (a_n^* \cos n\theta + b_n^* \sin n\theta) \right\} \cos k\theta \, d\theta \end{aligned}$$

$$=r_0^k a_k^*.$$

Similarly, $b_k(0)=r_0^k b_k^*$. Hence (3.5) becomes

$$\sum_{k=1}^{\infty} k r_0^k \left(a_k^* \int_0^{2\pi} h_\theta^*(r_0, \theta)\lambda(\theta)\cos k\theta\, d\theta \right.$$
$$\left. + b_k^* \int_0^{2\pi} h_\theta^*(r_0, \theta)\lambda(\theta)\sin k\theta\, d\theta \right) = 0 \tag{3.6}$$

But

$$h_r^*(r_0, \theta) = \sum_{k=1}^{\infty} k r_0^{k-1}(a_k^* \cos k\theta + b_k^* \sin k\theta)$$

Since the last series converges uniformly and since both h_θ^* and λ are analytic functions of θ, we can integrate this series term-by-term after taking the inner product with $r_0 h_\theta^*(r_0, \theta)\lambda(\theta)$ and get

$$\int_0^{2\pi} r_0 h_r^*(r_0, \theta) h_\theta^*(r_0, \theta)\lambda(\theta) d\theta = 0$$

Since $\lambda(\theta)$ can be $\sin k\theta$ and $\cos k\theta$, $k=1, 2, \cdots$, it follows that all Fourier coefficients of the function

$$r_0 h_r^*(r_0, \ ) h_\theta^*(r_0, \ )$$

are zero. But this is an analytic function of θ, so that we have

$$r_0 h_r^*(r_0, \theta) h_\theta^*(r_0, \theta) \equiv 0.$$

Since r_0 is also arbitrary, we conclude that h^* must satisfy the condition

$$r h_r(r, \theta) h_\theta(r, \theta) \equiv 0 \tag{3.7}$$

in B.

We now show that the equation (3.7) implies $E=G$ and $F=0$ from which we see that h^* satisfies the condition (2) of the problem.

Let $f=(f^1, f^2, \cdots, f^n)$ be the holomorphic function whose real part is h^* (componentwise) and let $\bar{h}^*$ be the harmonic conjugate of h^* which vanishes at the origin. Then h^* and $\bar{h}^*$ satisfy the Cauchy-Riemann equations

$$h_u^* = \bar{h}_v^*, \quad h_u^* = -h_v^*$$

and the derivative of f is given by

$$f'(w) = h_u^*(u, v) + i\bar{h}_u^*(u, v)$$
$$= h_u^*(u, v) - i h_v^*(u, v).$$

Thus

$$(f')^2 = (h_u^{*2} - h_v^{*2}) - 2i h_u^* h_v^*$$
$$= (E-G) - 2iF \tag{3.8}$$

Now in polar coordinate the Cauchy-Riemann equation becomes

$$r h_r^* = \bar{h}_\theta^*, \quad h_\theta^* = -r\bar{h}_r^*$$

and the derivative of f is given by

$$f' = e^{-i\theta}(h_r^* + i\bar{h}_r^*)$$

Multiplying by $w=re^{i\theta}$, we have

$$wf' = r(h_r^* + i\bar{h}_r^*)$$

so that

$$w^2 f'^2(w) = r^2(h_r^{*2} - \bar{h}_r^{*2} + 2i h_r^* \bar{h}_r^*).$$

Since $r\bar{h}_r^* = -h_\theta^*$, it follows that

$$-2r h_r^*(r, \theta) h_\theta^*(r, \theta) = \operatorname{Im} w^2 f'^2(w).$$

But $w_2f'_2(w)$ is holomorphic in B and $rh_1^*h_2^*=0$. So $w^2f'^2$ must be constant which vanishes at $w=0$. Hence $w^2f'^2(w)\equiv 0$ so that we have $f'^2(w)\equiv 0$. Thus, from (3.8), we get $E=G$ and $F=0$ in B.

The fact that $h^*|\partial B$ is a topological representation of Γ follows from the following well known theorem: thus the surface h^* with the minimum Dirichlet integral solves our existence problem.

THEOREM 4. *Suppose* ζ *is a surface which satisfies conditions* (1) *and* (2) *of the problem we set. If* $\zeta|\partial B$ *is a continuous map of* ∂B *onto* Γ *which carries three distinct points of* ∂B *into three distinct points of* Γ, *then* $\zeta|\partial B$ *is one-to-one: i.e. is a topological representation of* Γ.

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