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ON CLASSES OF NON-NORMAL OPERATORS

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1. Introduction

Let $B(H)$ denote the set of all bounded linear operators acting on the complex Hilbert space H. Let $R(T, z)=(T-z)^{-1}$, be the resolvent of T at z. Let $\sigma(T)$, $co\ \sigma(T)$, $\overline{W(T)}$, $r(T)$ and $\omega(T)$ respectively denote the spectrum, the convex hull of the spectrum, the closure of the numerical range, the spectral radius and the numerical radius of an operator T. For an operator T, the following inequalities hold; $\omega(T)\leqq||T||\leqq 2\omega(T)$. An operator T is *normaloid* [3] if $\omega(T)=||T||$, *spectraloid* [3] if $r(T)=\omega(T)$. We shall introduce a new class of operators as follows: An operator T is *of class* N_2 (denote $T\in N_2$) if $||T||=2\omega(T)$. In a recent paper [2], Furuta introduced new classes of operators: $T\in S$ if $||R(T,z)||=1/d(z, \sigma(T))$ for all $z\notin\overline{W(T)}$ and $T\in P$ if $\overline{W(T)}=\sigma(T)$. An operator T *satisfies the sequential* G_1 *property* (denote $T\in G_s$) [4] if for every $z\in\partial\sigma(T)$, the boundary of $\sigma(T)$, there exists a sequence $\{z_n\}$ in $\rho(T)$, the resolvent set of T, such that (i) $z_n\to z$ (ii) $||R(T,z_n)||=1/d(z_n, \sigma(T))$ for all n. Let π be the quotient map from $B(H)$ onto the Calkin algebra $B(H)/C(H)$ where $C(H)$ denotes the set of all compact operators in $B(H)$. An operator $T\in B(H)$ is *esssentially* N_2, *essentially* P, *essentially* S and *essenitally* G_s if $\pi(T)$ is an element of N_2, P, S and G_s respectively. We denote each of these sets by $e(N_2)$, $e(P)$, $e(S)$ and $e(G_s)$ respectively. Let $\sigma_e(T)$, $W_e(T)$, $r_e(T)$ and $\omega_e(T)$ denote the essential spectrum, the essentially numerical range, the essentially spectral radius and the essentially numerical radius of an operator T respectively.

In this paper, we obtain sufficient conditions for an operator to be in class N_2, class $e(P)$, class $e(S)$ and class G_s and discuss some topological properties of these classes.

2. Constructions

In this section, we give general methods to construct operators of class N_2, $e(N_2)$, $e(P)$, $e(S)$, G_s and $e(G_s)$. Methods to construct operators of class P and class S are given by Furuta [2]. It is easily shown that class N_2 and class S are closed under $\oplus$ operation.

THEORM 2. 1. *If A is any operator and* $B\in N_2$ (resp. $e(N_2)$) *with* $||A||\leqq\omega(B)$ (resp. $||\pi(A)||\leqq\omega_e(B)$), *then* $T=A\oplus B\in N_2$ (resp. $e(N_2)$).

Proof. $||T||=\max\{||A||, ||B||\}=||B||=2\omega(B)$. On the other hand, $\omega(T)=\max\{\omega(A), \omega(B)\}=\omega(B)$, Hence $||T||=2\omega(T)$, so that $T\in N_2$. The proof where $B\in e(N_2)$ with $||\pi(A)||\leqq\omega_e(B)$ is smilar.

THEORM 2.2. *If $T\in e(N_2)$, then there is a compact operator K such that $T+K\in N_2$.*

Proof. By[1], there is a compact operator K such that $W_e(T)=\overline{W(T+K)}$ and $||\pi(T)||=||T+K||$. Hence $||T+K||=||\pi(T)||=2\omega(\pi(T))=2\omega_e(T)=2\omega(T+K)$, so that $T+K\in N_2$.

We have the following: if T is essentially spectraloid, then there is a compact operator K such that $T+K$ is spectraloid by a similar proof to Theorem 2.2.

Using methods of contructing operators of class P and class S [2], we obtain the following results: (a) If A is any operator and $B\in e(P)$ with $W_e(A)\subset\sigma_e(B)$, then $T=A\oplus B\in e(P)$, and

(b) If A is any operator and $B\in e(S)$ such that $d(z, \sigma_e(B))\leqq d(z, W_e(A))$ for all $z\notin W_e(B)$, then $T=A\oplus B\in e(S)$.

THEOREM 2.3. *If A is any operator and $B\in G_s$ (resp. $e(G_s)$) with $\overline{W(A)}\subset\sigma(B)$ (resp. $W_e(A)\subset\sigma_e(B)$), then $T=A\oplus B\in G_s$ (resp. $e(G_s)$).*

Proof. We have $\sigma(T)=\sigma(A)\cup\sigma(B)=\sigma(B)$. For each $z\in\partial\sigma(T)$, there exists a sequence $\{z_n\}$ in $\rho(T)$ such that (i) $z_n\to z$ (ii) $||R(B, z_n)||=1/d(z_n, \sigma(B))$ for all n. Hence $||R(T, z_n)||=\max\{||R(A, z_n)||, ||R(B, z_n)||\}=\max\{||R(A, z_n)||, 1/d(z_n, \sigma(B))\}=1/d(z_n, \sigma(B))=1/d(z_n, \sigma(T))$, so that $T=A\oplus B\in G_s$. The proof where $B\in e(G_s)$ with $W_e(A)\subset\sigma_e(B)$ is similar.

3. Topological properties

In this section, we assume that $B(H)$ has the uniform operator topology. It is easily shown that (a) $T\in N_2$ (resp. P and G_s) implies $T^*\in N_2$ (resp. P and G_s), and (b) N_2 is an arcwise connected, closed subset of $B(H)$.

THEOREM 3.1. *P is an arcwise connected, closed subset of $B(H)$.*

Proof. Since $\sigma(aT)=a\sigma(T)$ and $W(aT)=aW(T)$, $T\in P$ implies $aT\in \rho$ for every complex number a. We see that the ray in $B(H)$ through T is contained in P. Therefore P is arcwise connected. Let $\{T_n\}$ be a sequence in P such that $T_n\to T$ in $B(H)$. By [3 Problem 175], $\overline{W(T_n)}\to\overline{W(T)}$. On the other hand, $\sigma(T_n)\to\sigma(T)$ by [5] since $P\subset G_1$. Hence $\overline{W(T)}=\sigma(T)$.

From the above facts, it follows that $e(N_2)$ and $e(P)$ are closed subsets of $B(H)$.

THEOREM 3.2. *S is an arcwise connected subset of class of convexoid operators and strongly dense in $B(H)$.*

Proof. It is obvious that S is arcwise connected. Let $T\in B(H)$ be arbitrary and $\{x_1, \cdots\cdots, x_n\}\subset H$ and let M be the space generated by $\{x_1, \cdots\cdots, x_n\}$. We define the operator A on H as follows; $Ax_i=Tx_i$, $i=1, \cdots\cdots, n$ and $Ax=0$ if $x\in M^\perp$. We consider a normal operator N on $M^\perp$ such that $d(z, \sigma(N))\leqq d(z, W(A))$ for all $z\notin \overline{W(N)}$. By [2], $A\oplus N\in S$ and since it coincides with T on x_i, the strong density is clear.

In a similar way, we obtain that G_s is an arcwise connected subset of $B(H)$ and strongly dense in $B(H)$.

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