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REMARKS ON THE CARISTI-KIRK FIXED POINT THEOREM

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1. Introduction

In attempting to improve the Caristi-Kirk fixed point theorem, Kirk has raised the question of whether f continues to have a fixed point if we replace $d(x,fx)$ by $d(x, fx)^p$ where $p>1$ in the following theorem (cf. [3]).

THEOREM A (Caristi-Kirk [2]). *Let (M, d) be a complete metric space, $f: M\rightarrow M$ an arbitrary map, and $\phi: M\rightarrow \boldsymbol{R}^+$ a lower semicontinuous function. If $d(x, fx)\leq \phi(x)-\phi(fx)$ for all x in M, then f has a fixed point in M.*

In this paper, we first give an example which shows that Kirk's problem is not affirmative even if ϕ and f are continuous.

In section 3, we consider certain circumstances where Kirk's problem is valid, and, consequently obtain generalizations of results of Caristi [2], Ekeland [5], and Park [7].

Actually, Kasahara [6] and Siegel [8] obtained the following generalization of the Caristi-Kirk theorem.

THEORM B. *Let (M, d) be a complete metric space, and $\phi: M\rightarrow \boldsymbol{R}^+$ a lower semicontinuous function. Then the family*

$$F=\{f: M\rightarrow M \mid d(x, fx)\leq \phi(x)-\phi(fx) \text{ for } x\in M\}$$

has a common fixed point.

Note that F is not empty since $1_M\in F$. In fact, such a common fixed point in Theorem B is a d-point in the following theorem of Ekeland [4], [5].

THEOREM C. *Every lower semicontinuous function ϕ from a complete metric space (M, d) into $\boldsymbol{R}^+$ has a d-point q in M, that is, we have*

$$\phi(q)-\phi(x)<d(q,x)$$

for every other point x in M.

2. An example

We give an example showing that Kirk's problem is not affirmative when $p>1$ even if ϕ and f are continuous.

Let $M=\boldsymbol{R}$ and $\phi: \boldsymbol{R}\rightarrow \boldsymbol{R}^+$ such that

$$\phi(x)=\begin{cases} 2x+3 & \text{if } x\geq -1 \\ -\dfrac{1}{x} & \text{if } x\leq -1. \end{cases}$$

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Then ϕ is continuous. Define $f: \boldsymbol{R} \to \boldsymbol{R}$ by $fx = x - \varepsilon(x)$ with sufficiently small $\varepsilon(x)$, $0 < \varepsilon(x) < \frac{1}{4}$ so that f satisfies $d(x, fx)^p \leq \phi(x) - \phi(fx)$ for $x \in \boldsymbol{R}$ with the usual metric d on $\boldsymbol{R}$ where $p > 1$

In fact, $d(x, fx)^p = \varepsilon(x)^p$ and

(i) if $x \geq -\frac{3}{4}$, then $\phi(x) - \phi(fx) = 2\varepsilon(x)$,

(ii) if $x < -\frac{3}{4}$, then we have

$$\phi(x) - \phi(fx) \geq -\frac{1}{x} + \frac{1}{x - \varepsilon(x)} = \frac{\varepsilon(x)}{x(x - \varepsilon(x))}.$$

We can choose sufficiently small $\varepsilon(x) > 0$ so that $\varepsilon(x)^{p-1}(|x| + \varepsilon(x))^2 \leq 1$, and hence $\varepsilon(x)^p \leq \varepsilon(x)/x(x - \varepsilon(x))$, for each fixed $x < -\frac{3}{4}$. Therefore in any case, we can choose $\varepsilon(x)$ so that f is continuous and $d(x, fx)^p \leq \phi(x) - \phi(fx)$ holds. However f has no fixed point.

3. Main results

It is well-known that Theorems A and C are equivalent (Brézis-Browder [1]). This can be expressed more explicitly as follows by combining Theorems B and C:

THEOREM 1. *Let (M, d) be a metric space, $\phi: M \to \boldsymbol{R}^+$ an arbitrary function. Let F be the family of all selfmaps of M such that for each $x \in M$,*

$$(*) \qquad d(x, fx)^p \leq \phi(x) - \phi(fx)$$

where $p > 0$. Then $q \in M$ is a common fixed point of F iff q satisfies $\phi(q) - \phi(x) < d(q, x)^p$ for every other point x in M.

Proof. Sufficiency is obvious. To see the necessity, suppose there exists a $y \in M$ with $y \neq q$ such that $\phi(q) - \phi(y) > d(q, y)^p$. Define $f: M \to M$ such that $fq = y$ and $fx = x$ for $x \neq q$. Then $f \in F$ and q is not a fixed point of f.

REMARK. In Theorem 1, if M is complete and ϕ is lower semicontinuous, and if $0 < p \leq 1$, then ϕ has a point q in M satisfying $\phi(q) - \phi(x) < d(q, x)^p$ for each other point x in M. This extends Ekeland's Theorem C. To prove this, consider a new metric ρ on M with $\rho(x, y) = d(x, y)/(1 + d(x, y))$ which is equivalent to the original metric d.

Let M be a complete metric space and $\phi: M \to \boldsymbol{R}^+$ a lower semicontinuous function. Let D be the set of all d-points of ϕ in M. We say that ϕ has a minimal d-point q in M if $q \in D$ and $\phi(q) = \inf_{q' \in D} \phi(q')$. Note that if ϕ has a finite number of d-points, then a minimal d-point of ϕ always exists.

LEMMA 1. *Let (M, d) be a complete metric space and $\phi: M \to \boldsymbol{R}^+$ a lower semicontinuous function. Then q is a minimal d-pojnt of ϕ in M iff $\inf_{x \in M} \phi(x) = \phi(q)$.*

Proof. If $\phi(q) = \inf_{x \in M} \phi(x)$, then q is clearly a minimal d-point of ϕ in M. Conversely, suppose that q is a minimal d-point of ϕ in M and $\inf_{x \in M} \phi(x) < \phi(q)$. Let

r be a number such that $\inf_{x\in M}\phi(x)<r<\phi(q)$ and $N=\{x\in M \mid \phi(x)<r\}$. Since ϕ is lower semicontinuous, N is closed in M. By Theorem C, ϕ has a point q' in N such that $\phi(q')-\phi(x)<d(q', x)$ for every other point x in N. Let $y\notin N$. Then $\phi(y)>r$ and $\phi(q')\leq r$ give $\phi(q')-\phi(y)<d(q',y)$. Hence q' is another d-point in M and $\phi(q')<\phi(q)$, which leads a contradiction.

LEMMA 2. *Every lower semicontinuous function ϕ from a compact metric space M into $\boldsymbol{R}^+$ has a minimal d-point in M.*

Proof. Let $\inf_{x\in M}\phi(x)=r$. Then there exists a sequence $\{x_i\}$ in M such that $\phi(x_i)\to r$. Since M is compact, we may assume that $\{x_i\}$ converges to some point q in M. Then $r=\lim\phi(x_i)\geq\phi(q)$. Hence $\phi(q)=r$ and by Lemma 1, q is a minimal d-point of ϕ in M.

THEOREM 2. *Let (M,d) be a complete metric space and $\phi: M\to\boldsymbol{R}^+$ a lower semicontinuous function. Let G be the family of all selfmaps of M satisfying $(*)$.*

(i) *If $0<p\leq 1$, then F has a common fixed point.*

(ii) *If ϕ has a minimal d-point q in M, then q is a common fixed point of F.*

Proof. (i) Since $0<p\leq 1$, $\rho(x,y)=d(x,y)/(1+d(x,y))\leq\{d(x,y)/(1+d(x,y))\}^p\leq d(x,y)^p$ for all $x,y\in M$. Hence we have $\rho(x,fx)\leq\phi(x)-\phi(fx)$, $x\in M$ and so we can apply Theorems 1 and C to get the desired result.

(ii) By Lemma 1, $\phi(q)=\inf_{x\in M}\phi(x)$. Then clearly $\phi(q)-\phi(x)<d(q,x)^p$ for every other point x in M. Therefore by Theorem 1, q is a common fixed point of F.

REMARK. Note that Theorem 2 (i) also extends Theorems A and B. Since Theorems A and C are equivalent, Theorem 2 (i) also extends Ekeland's Theorem C.

Also note that Theorem 2 (ii) says that if M is compact or ϕ has finitely many d-points, then F has a common fixed point by Lemmas 1 and 2.

THEOREM 3. *Let (M, d) be a metric space and f a continuous selfmap of M. Let $\phi: M\to\boldsymbol{R}^+$ be an arbitrary function satisfying $(*)$.*

(i) If $x\in M$, *then any cluster point of the iteration $\{f^n x\}_{n=0}^{\infty}$ is a fixed point of f.*

(ii) *If M is complete and $0<p\leq 1$, then $\{f^n x\}$ converges to some fixed point of f for every $x\in M$.*

Proof. (i) Since $\{\phi(f^n x)\}$ is decreasing and bounded below, $\lim d(f^n x, f^{n+1}x)=0$. Let q be a cluster point of $\{f^n x\}$ and let $\{f^{n_i}x\}$ be a subsequence of $\{f^n x\}$ converging to q. Since

$$d(q, fq)<d(q, f^{n_i}x)+d(f^{n_i}x, f^{n_i+1}x)+d(f^{n_i+1}x, fq)$$

and all terms of the right hand side converge to 0, we have $fq=q$.

(ii) We know that the new metric ρ with $\rho(x, y)=d(x,y)/(1+d(x,y)$ is equivalent to the original metric d and $\rho(x, fx)<\phi(x)-\phi(fx)$. Since $\{\phi(f^n x)\}$ is decreasing and bounded below, $\{f^n x\}$ is a Cauchy sequence in M for every x in M with the metric ρ and hence with d. Therefore, there is a point q in M such that $f^n x\to q$. Since f is continuous, we have $fq=q$.

Note that in Theorem 3, we did not assume the lower semicontinuity of ϕ.

THEOREM 4. *Let X be a nonempty set. (M, d) a complete metric space, and f,g:*

$X \to M$ maps such that

(a) *f is surjective, and*

(b) *there exists a lower semicontinuous function $\phi : M \to \mathbf{R}^+$ satisfying*

$$d(fx, gx)^p \leq \phi(fx) - \phi(gx)$$

for each $x \in X$, where $p > 0$.

(i) *If $0 < p \leq 1$, then f and g have a coincidence.*

(ii) *If ϕ has a minimal d-point in M, then f and g have a coincidence.*

Proof. In any case of (i) and (ii), by the same argument in the proof of Theorem 2, we have a point q in M such $\phi(q) - \phi(y) < d(q, y)^p$ for every other point y in M. Let $x \in f^{-1}q$. Suppose $fx \neq gx$. Then we have

$$\phi(fx) - \phi(gx) = \phi(q) - \phi(gx) < d(q, gx)^p = d(fx, gx)^p,$$

which is a contradication.

REMARK. Theorem 4 (i) includes Proposition 7 in [7].

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